

Riemannian Supergeometry

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Abstract

Motivated by Zirnbauer [Zir 1996], we develop a theory of Riemannian supermanifolds up to a definition of Riemannian symmetric superspaces. Various fundamental concepts needed for the study of these spaces both from the Riemannian and the Lie theoretical viewpoint are introduced, e.g. geodesics, isometry groups and invariant metrics on Lie supergroups and homogeneous superspaces.

1 Introduction

Although there exists a theory of differential geometry of supermanifolds¹ – see Deligne and Morgan [DelMor 1999], Schmitt [Schm 1984] or Varadarajan [Var 2004] – a notion of Riemannian metric for supermanifolds only seldom occurs in the literature. More precisely, beyond the existence of a Levi-Civita connection – see Monterde and Sánchez-Valenzuela [MonSan 1996] – no general theory of Riemannian supermanifolds is available.

The motivation for the development of such a theory came from the physicists: in 1996, Zirnbauer [Zir 1996] defined Riemannian symmetric superspaces to be a quotient of complex Lie supergroups, together with some distinguished Riemannian symmetric space embedded into the underlying manifold.

Our aim is to give a definition of these objects similar to the standard theory, namely as Riemannian supermanifolds with a symmetry property, and afterwards recognize them as special homogeneous superspaces. But before that, various fundamental concepts have to be introduced and studied, e.g. geodesics, isometry groups and invariant metrics on Lie supergroups and homogeneous superspaces.

It is to be mentioned that the non-linear theory developed in this thesis already has some infinitesimal counterpart in the mathematical literature; for example, Cortés [Cor 2003] defines a notion of infinitesimal pseudo-Riemannian symmetric superspace, and Serganova [Ser 1983] lists up involutive automorphisms of the simple Lie superalgebras over \mathbf{R} and \mathbf{C} .

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¹There exist various versions of supergeometry; the formalism we use is the sheaf-theoretic approach by Berezin, Kostant and Leites [Ber 1987], [Kost 1975], [Lei 1980].

In the sections 2 and 3 we review the basics of differential supergeometry and Lie supergroups; we assume some familiarity with linear superalgebra, see e.g. Varadarajan [Var 2004] or Deligne and Morgan [DelMor 1999].

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2 Foundations of Supergeometry

2.1 Supermanifolds and their Morphisms

The model in the category of supermanifolds is the space $\mathbf{R}^{n|m}$, which is by definition the ringed space consisting of the topological space \mathbf{R}^n and the sheaf of super \mathbf{R} -algebras $\mathcal{C}^\infty \otimes \Lambda_{\mathbf{R}}[\xi_1, \dots, \xi_m]$.

A *supermanifold* (*graded manifold*) of dimension $n|m$ is a ringed space $M = (|M|, \mathcal{O}_M)$, where $|M|$ is a topological space (Hausdorff, countable base) and the structural sheaf \mathcal{O}_M is a sheaf of super \mathbf{R} -algebras with unity, locally isomorphic to $\mathbf{R}^{n|m}$. Sections of the structural sheaf are referred to as *superfunctions* on M ; if there is no danger of confusion, we simply call them *functions*.

Let M and N be supermanifolds. A *morphism* $\Phi : M \rightarrow N$ is a morphism of ringed spaces: $\Phi = (\phi, \phi^*)$, where ϕ is a continuous map between the underlying topological spaces and $\phi^* : \mathcal{O}_N \rightarrow \phi_* \mathcal{O}_M$ is a morphism of sheaves of \mathbf{R} -super algebras with unity. The morphism is *not* determined by the map between the topological spaces; nevertheless, the map on global sections $\phi_N^* : \mathcal{O}_N(N) \rightarrow \phi_* \mathcal{O}_M(N) = \mathcal{O}_M(M)$ determines the whole morphism, i.e. ϕ and ϕ^* , cf. [Kost 1975], p. 208.

The nilpotent functions define an ideal sheaf J of \mathcal{O}_M , and the ringed space $M_{\text{red}} := (|M|, \mathcal{O}_M/J)$ is a differentiable manifold. We call it the *reduced* or *underlying manifold* or the *support* of M . The quotient map $\mathcal{O}_M \rightarrow \mathcal{O}_M/J$ defines a morphism $M_{\text{red}} \rightarrow M$ sending a superfunction f on M to a smooth function \tilde{f} on the reduced manifold. The *value* of a superfunction f at some point $p \in M_{\text{red}}$ is defined to be $\tilde{f}(p)$, which coincides with the unique real number λ such that $f - \lambda$ is not invertible as an element of the stalk $\mathcal{O}_{M,p}$; sometimes, we simply write $f(p)$ for the value of f of p although this might be misleading – since f is not determined by all its values, associating to f the function sending p to $f(p)$ is not an injective mapping. Hence, this does not provide us with a realization of \mathcal{O}_M as a sheaf of ordinary real-valued functions.

If $U \subset M_{\text{red}}$ is such that $\mathcal{O}_M(U) = \mathcal{C}^\infty(U) \otimes \Lambda_{\mathbf{R}}[\xi_1, \dots, \xi_m]$ and coordinates x_i of the reduced manifold on U are given, then we call (x_i, ξ_α) *coordinates of M on U* . Note that our convention is to give roman indices for even (here: the *even coordinates* x_i) and greek indices for odd objects (here: the *odd coordinates* ξ_α) – this will be convenient for notation purposes. However, if no distinction between even and odd coordinates is necessary, we simply write (η_i) .

A supermanifold of dimension $n|0$ is an ordinary differentiable manifold of dimension n , so at any time we may (and should) test the soundness of our theory by setting $m = 0$.

One way of constructing examples of supermanifolds is the following: If M is an ordinary differentiable manifold and $E \rightarrow M$ a vector bundle, then $(M, \Gamma(\Lambda E))$ is a supermanifold, where $\Gamma(\Lambda E)$ is the sheaf of sections of the exterior bundle $\Lambda E \rightarrow M$. Note that $\Gamma(\Lambda E)$ possesses a natural \mathbf{Z} -grading; in regarding it as the structural sheaf of a supermanifold, we retain only the induced \mathbf{Z}_2 -grading. Although the Theorem of Batchelor [Bat 1979] asserts that every supermanifold over a differentiable manifold M is isomorphic to one constructed in this way from some vector bundle over M , we prefer the definition given above since this isomorphism is non-canonical.

2.2 Tangent Sheaf and Vector Fields

For a super \mathbf{R} -algebra A , we give the endomorphisms $\text{End } A$ of A the structure of a super vector space via the natural grading

$$(\text{End } A)_0 = \{\varphi \in \text{End } A \mid \varphi(A_0) \subset A_0, \varphi(A_1) \subset A_1\},$$

$$(\text{End } A)_1 = \{\varphi \in \text{End } A \mid \varphi(A_0) \subset A_1, \varphi(A_1) \subset A_0\}.$$

Recall that a homogeneous element $\varphi \in \text{End } A$ is a homogeneous derivation if

$$\varphi(ab) = \varphi(a) \cdot b + (-1)^{|\varphi||a|} a \cdot \varphi(b) \quad (2.1)$$

for all $a, b \in A$, where for a homogeneous element x of some graded object, $|x| \in \{0, 1\}$ denotes the parity of x . An element $\varphi \in \text{End } A$ is a derivation if its homogeneous components are homogeneous derivations.

If M is a supermanifold of dimension $n|m$, we define

$$\mathcal{T}_M(U) := \text{Der}(\mathcal{O}_M(U)),$$

the $\mathcal{O}_M(U)$ -super module of derivations of $\mathcal{O}_M(U)$. For $V \subset U \subset M$ there is a natural restriction map $\mathcal{T}_M(U) \rightarrow \mathcal{T}_M(V)$ turning \mathcal{T}_M into a sheaf of \mathcal{O}_M -super modules, see [Schm 1984], p.160. The \mathcal{O}_M -module \mathcal{T}_M is locally free of dimension $n|m$, cf. [DelMor 1999], §3.3. The sections of \mathcal{T}_M are called *vector fields*. We will refer to \mathcal{T}_M itself either as the *tangent sheaf* or the *tangent bundle* of the supermanifold M ; this is not too much abuse of language, as is pointed out in [DelMor 1999], §3.4

On $\mathcal{T}_M(U)$ we have a bracket $[\cdot, \cdot]$ defined by

$$[X, Y]f := X(Yf) - (-1)^{|X||Y|} Y(Xf).$$

It satisfies the *graded Jacobi identity*

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{|X||Y|} [Y, [X, Z]] \quad (2.2)$$

and thus turns $\mathcal{T}_M(U)$ into a Lie superalgebra.

For every point p of M , the *tangent space* $T_p M$ of M at p is defined to be the space of derivations $\varphi : \mathcal{O}_{M,p} \rightarrow \mathbf{R}$, i.e.

$$\varphi(fg) = \varphi(f)g(p) + (-1)^{|\varphi||f|} f(p)\varphi(g),$$

where $\mathcal{O}_{M,p}$ is the stalk of \mathcal{O}_M at p . For $p \in U$, there is a natural mapping $\mathcal{T}_M(U) \rightarrow T_p M$ sending a vector field X to its value X_p at p ; nevertheless, a vector field is not determined by its values at all points. The tangent spaces are the fibres of a bundle $TM \rightarrow M_{\text{red}}$ of rank $n+m$ which canonically splits as the direct sum of the tangent bundle of the reduced manifold, $TM_{\text{red}} \rightarrow M_{\text{red}}$, and a bundle $(TM)_1 \rightarrow M_{\text{red}}$ with the odd parts of the tangent spaces as fibres. For a vector field X on M , we denote by \tilde{X} the associated section of TM .

The *cotangent bundle* of a supermanifold M is by definition the dual Ω_M^1 of \mathcal{T}_M . As in [DeMor 1999], we will write the duality pairing between the tangent and cotangent bundle as

$$\langle \cdot, \cdot \rangle : \mathcal{T}_M \otimes_{\mathcal{O}_M} \Omega_M^1 \rightarrow \mathcal{O}_M$$

with $\langle uX, v\omega \rangle = (-1)^{|X||v|} uv \langle X, \omega \rangle$ for $u, v \in \mathcal{O}_M$. Then, as usual, we define $d : \mathcal{O}_M \rightarrow \Omega_M^1$ by

$$\langle X, df \rangle = Xf.$$

A *vector field along a morphism* $\Phi : M \rightarrow N$ on $U \subset N$ is a morphism of super vector spaces

$$X : \mathcal{O}_N(U) \rightarrow \phi_* \mathcal{O}_M(U) = \mathcal{O}_M(\phi^{-1}(U))$$

such that its homogeneous components satisfy the derivation property

$$X(fg) = (Xf) \cdot \phi^*(g) + (-1)^{|X||f|} \phi^*(f) \cdot (Xg) \quad (2.3)$$

for all $f, g \in \mathcal{O}_N(U)$, cf. [CarFig 1997]. The set of such vector fields will be denoted by $\text{Der}_\Phi(U)$ and the corresponding sheaf of vector fields along Φ by $\mathcal{T}_\Phi := \text{Der}_\Phi$.

There are two standard ways of constructing vector fields along Φ : If X is a vector field on N , then

$$\hat{X} := \phi^* \circ X$$

is a vector field along Φ ; a vector field Y on M yields one by attaching ϕ^* on the other side: we define

$$d\Phi(Y) := Y \circ \phi^*. \quad (2.4)$$

If Φ is a diffeomorphism, we often use the same notation for the vector field $d\Phi(Y) := (\phi^{-1})^* \circ Y \circ \phi^*$ on N .

If $\Phi : M \rightarrow N$ is a morphism of supermanifolds, then we have induced linear maps $d_p \Phi : T_p M \rightarrow T_{\phi(p)} N$. We call Φ an *immersion at p* if $d_p \Phi$ is injective, and a *submersion at p* if $d_p \Phi$ is surjective. See [Var 2004], p.148 for the local structure of immersions and submersions.

The sheaf \mathcal{T}_Φ is a locally free sheaf of $\phi_* \mathcal{O}_M$ -modules over N of the same rank as \mathcal{T}_N . More precisely, if (x_i, ξ_α) are local coordinates on $U \subset N$, then $(\phi^* \circ \partial_{x_i} = \hat{\partial}_{x_i} = \hat{\partial}_i, \phi^* \circ \partial_{\xi_\alpha} = \hat{\partial}_{\xi_\alpha} = \hat{\partial}_\alpha)$ is a basis of $\mathcal{T}_\Phi(U)$: any $X \in \mathcal{T}_\Phi(U)$ can be written uniquely as

$$X = \sum f_i \hat{\partial}_{x_i} + \sum g_\alpha \hat{\partial}_{\xi_\alpha}$$

with $f_i, g_\alpha \in \phi_* \mathcal{O}_M(U)$ [CarFig 1997]. Note that

$$d\Phi(Y) = Y \circ \phi^* = \sum Y(\phi^* x_i) \cdot \hat{\partial}_{x_i} + \sum Y(\phi^* \xi_\alpha) \cdot \hat{\partial}_{\xi_\alpha} \quad (2.5)$$

for all vector fields Y on M .

3 Lie Supergroups

3.1 Lie Supergroups and their Lie Superalgebras

A Lie supergroup is a group object in the category of supermanifolds, i.e. a supermanifold G together with morphisms $m : G \times G \rightarrow G$, $i : G \rightarrow G$ and $1 : \mathbf{R}^{0|0} \rightarrow G$ representing the multiplication map, the inverse map and the unit element such that the usual group axioms are satisfied. The associativity law for example reads

$$m \circ (\text{id}_G \times m) = m \circ (m \times \text{id}_G),$$

cf. Varadarajan [Var 2004].

The reduced morphisms turn the reduced manifold G_{red} into a Lie group. A Lie supergroup H is a *Lie subsupergroup* of a Lie supergroup G if H_{red} is a Lie subgroup of G_{red} and the inclusion map of H into G is a morphism that is an immersion everywhere.

In classical Lie theory, a vector field X on a Lie group G is left-invariant if and only if $X_{gh} = dl_g(X_h)$ for all $g, h \in G$, where l_g is left translation by g . We have to reformulate this in a way not using the elements of G before generalizing the definition. A short calculation shows that it is equivalent to the condition

$$(I \otimes X) \circ m^* = m^* \circ X, \quad (3.1)$$

where $I \otimes X$ is a vector field on $G \times G$, defined in the obvious way by acting only on the second component. This is now taken as the definition of *left-invariant vector field* on a Lie supergroup G ; analogously, we say that X is *right-invariant* if

$$(X \otimes I) \circ m^* = m^* \circ X. \quad (3.2)$$

The *Lie (super)algebra of G* is by definition the Lie superalgebra \mathfrak{g} of all left-invariant vector fields on G . The usual isomorphism between the Lie algebra of G and the tangent space of G in the identity is still valid, as is proven in [Var 2004], p.276f: The map $\mathfrak{g} \rightarrow T_e G; X \mapsto X_e$ is a linear isomorphism of super vector spaces. The converse map is also given: If $\tau \in T_e G$,

$$X_\tau := (I \otimes \tau) \circ m^* \quad (3.3)$$

is the left invariant vector field with $(X_\tau)_e = \tau$; here, for a germ f at $g \in G$, $m^* f$ is considered as a germ at (g, e) so that $I \otimes \tau$ can be applied.

We thus see that a Lie supergroup comes along with two objects we are more familiar with: its underlying Lie group and its Lie superalgebra. It would be nice if a Lie supergroup was already determined by this data.

3.2 Harish-Chandra Pairs

A *Harish-Chandra pair* is a pair (G_0, \mathfrak{g}) , consisting of a Lie group G_0 and a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with $\text{Lie}(G_0) = \mathfrak{g}_0$ and a representation Ad of G_0 on \mathfrak{g} such that

1. it extends the usual adjoint action of G_0 on its Lie algebra and

2. the differential of the action in the identity is equal to the Lie superbracket, restricted to $\mathfrak{g}_0 \times \mathfrak{g}$.

Note that if G_0 is connected, the first condition follows from the second.

It is clear how to associate a Harish-Chandra pair to a given Lie supergroup G : Take the underlying Lie group G_{red} , together with the Lie superalgebra \mathfrak{g} of G . The importance of the notion of Harish-Chandra pair results now from this functor being an equivalence of categories, cf. [DelMor 1999], §3.8. See [Kosz 1982] and [BagSta 2002] for the construction of the Lie supergroup associated to a Harish-Chandra pair.

3.3 Actions and Representations

An action of a Lie supergroup on a supermanifold is a morphism $G \times M \rightarrow M$ such that the usual axioms are satisfied.

In the language of Harish-Chandra pairs, such an action consists of an action of the reduced Lie group G_{red} on the supermanifold M , together with a morphism from \mathfrak{g} to the opposite to the Lie superalgebra of vector fields on M , which are compatible in the sense that the differential of the action of the reduced group at the identity agrees with the restriction to the Lie algebra \mathfrak{g}_0 , see [DelMor 1999], p. 80.

If an action $\rho : G \times M \rightarrow M$ is given, this morphism $\mathfrak{g} \rightarrow \mathcal{T}_M(M)^\circ$ is

$$X \mapsto (X_e \otimes I) \circ \rho^*; \quad (3.4)$$

note that it really is a morphism of Lie superalgebras, i.e.

$$([X, Y]_e \otimes I) \circ \rho^* = -[(X_e \otimes I) \circ \rho^*, (Y_e \otimes I) \circ \rho^*]. \quad (3.5)$$

Similarly, a *representation* of a Lie supergroup G on a super vector space V consists of representations of G_{red} and \mathfrak{g} on V such that the differential of the representation of G_{red} coincides with the even part of the representation of \mathfrak{g} .

3.4 Examples

In this section, we define those Lie supergroups and their Lie superalgebras that will be of importance for us in terms of their Harish-Chandra pairs. The adjoint action of the Lie group on the Lie superalgebra is always the standard one.

The *general linear supergroup* $\text{GL}(n|m)$ is defined to be the Lie supergroup associated to the Harish-Chandra pair

$$(\text{GL}(n) \times \text{GL}(m), \mathfrak{gl}(n|m)),$$

where $\mathfrak{gl}(n|m)$ is the Lie superalgebra consisting of block matrices $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right)$ with A, B, C and D real $n \times n$ -, $n \times m$ -, $m \times n$ - and $m \times m$ -matrices, respectively. The gradation is given by

$$\mathfrak{gl}(n|m)_0 = \left\{ \left(\begin{array}{c|c} A & 0 \\ \hline 0 & D \end{array} \right) \right\} \text{ and } \mathfrak{gl}(n|m)_1 = \left\{ \left(\begin{array}{c|c} 0 & B \\ \hline C & 0 \end{array} \right) \right\}$$

and the bracket is the usual (anti-)commutator: for homogeneous elements $X, Y \in \mathfrak{gl}(n|m)$ we define $[X, Y] := XY - (-1)^{|X||Y|}YX$.

The *special linear supergroup* $\mathrm{SL}(n|m)$ is the Lie subsupergroup of $\mathrm{GL}(n|m)$ with reduced group

$$\mathrm{SL}(n|m)_{\mathrm{red}} = \{(A, B) \in \mathrm{GL}(n) \times \mathrm{GL}(m) \mid \det A = \det B > 0\}$$

and Lie superalgebra

$$\mathfrak{sl}(n|m) = \{X \in \mathfrak{gl}(n|m) \mid \mathrm{str}(X) = 0\},$$

where the *supertrace* str of a matrix $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$ is given by $\mathrm{tr}(A) - \mathrm{tr}(D)$. Note that the reduced group of $\mathrm{SL}(n|m)$ is isomorphic to $\mathrm{SL}(n) \times \mathrm{SL}(m) \times \mathbf{R}$ via $(A, B, \lambda) \mapsto (e^{\frac{\lambda}{n}}A, e^{\frac{\lambda}{m}}B)$.

In the case of $n = m$, the identity matrix I_{2n} is an even element of the Lie superalgebra $\mathfrak{sl}(n|n)$ and generates a one-dimensional ideal; dividing this out, we obtain a Lie superalgebra denoted by $\mathfrak{psl}(n|n)$. The corresponding Lie supergroup $\mathrm{PSL}(n|n)$ is given by the Harish-Chandra pair

$$(\mathrm{SL}(n|n)_{\mathrm{red}}/\mathbf{R}, \mathfrak{psl}(n|n)).$$

The *orthosymplectic supergroup* $\mathrm{OSp}(n|2m)$ is the Lie subsupergroup of $\mathrm{GL}(n|2m)$ given by the Harish-Chandra pair

$$(\mathrm{O}(n) \times \mathrm{Sp}(m; \mathbf{R}), \mathfrak{osp}(n|2m)),$$

where

$$\mathfrak{osp}(n|2m) = \left\{ \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -B_2^t & C_1 & C_2 \\ B_1^t & C_3 & -C_1^t \end{array} \right) \mid A^t = -A, C_2^t = C_2, C_3^t = C_3 \right\}.$$

Here, the real symplectic group $\mathrm{Sp}(m; \mathbf{R})$ is the group of those transformations of \mathbf{R}^{2m} leaving invariant the standard symplectic form. The *special orthosymplectic supergroup* $\mathrm{SOSp}(n|2m)$ is the connected component of $\mathrm{OSp}(n|2m)$.

The *unitary supergroup* $\mathrm{U}(n|m)$ is the Lie supergroup associated to the Harish-Chandra pair

$$(\mathrm{U}(n) \times \mathrm{U}(m), \mathfrak{u}(n|m)),$$

where

$$\mathfrak{u}(n|m) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline -iB^* & C \end{array} \right) \mid A, B, C \text{ complex}, A^* = -A, C^* = -C \right\}.$$

4 Riemannian Supergeometry

4.1 Metrics

A *scalar superproduct* on a super vector space $V = V_0 \oplus V_1$ over a field K (we will be interested almost exclusively in the case $K = \mathbf{R}$) is a non-degenerate

graded-symmetric even K -bilinear form $\langle \cdot, \cdot \rangle : V \times V \rightarrow K$. Here, the condition of graded symmetry is supposed to mean

$$\langle X, Y \rangle = (-1)^{|X||Y|} \langle Y, X \rangle$$

for all homogeneous $X, Y \in V$. Since K is considered as a purely even object, i.e. $K_1 = 0$, being even means $\langle X, Y \rangle = 0$ for homogeneous X, Y of different parity. Note that a graded scalar product on V is the sum of a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_0$ on V_0 and a symplectic (i.e. non-degenerate alternating bilinear) form $\langle \cdot, \cdot \rangle_1$ on V_1 . In particular, the existence of a scalar superproduct on V forces V_1 to be even-dimensional.

Remark. Note that a scalar superproduct on a purely even vector space is *not* the same as a scalar product since we do not impose any kind of positivity – $\langle \cdot, \cdot \rangle_0$ may be indefinite. See also [Cor 2003].

We define a *graded Riemannian metric* on a supermanifold M as a graded-symmetric even non-degenerate \mathcal{O}_M -linear morphism of sheaves

$$\langle \cdot, \cdot \rangle : \mathcal{T}_M \otimes \mathcal{T}_M \rightarrow \mathcal{O}_M,$$

the non-degeneracy meaning that the mapping $X \mapsto \langle X, \cdot \rangle$ is an isomorphism $\mathcal{T}_M \rightarrow \Omega_M^1$. A supermanifold equipped with a graded Riemannian metric is called a *Riemannian supermanifold*.

For each $p \in M$ the morphism $\langle \cdot, \cdot \rangle$ defines a scalar superproduct $\langle \cdot, \cdot \rangle_p$ on the real super vector space $T_p M$. As always, this family of scalar superproducts does not determine the graded Riemannian metric unless the odd dimension of M is zero.

Let us do the usual consistency check: A graded Riemannian metric induces in a natural way a pseudo-Riemannian metric on the underlying manifold M_{red} : take $\langle \cdot, \cdot \rangle_{p,0}$ on $T_p M_{\text{red}} = (T_p M)_0$. Furthermore, on a usual differentiable manifold, the notion of graded Riemannian metric equals the notion of pseudo-Riemannian metric.

Remark. The choice of name for our metrics is justified by the fact that the natural metrics on Lie supergroups and homogeneous superspaces induced by the Killing form almost never have a definite sign, see 4.10. The class of metrics on supermanifolds such that the underlying manifold is Riemannian (and not only pseudo-Riemannian) seems to play only a minor role. See also [MonSan 1996] and [MonSan 1997].

We also remark that there is another way of defining metrics having the nice property that the positive linear combination of metrics again is a metric – which in our context obviously is not fulfilled. By passing to complexifications one may apply a definition of Tuynman in the context of cs manifolds, see [Tuy 2004], p. 188ff.

If a morphism $\Phi : M \rightarrow N$ and a graded Riemannian metric on N are given, we can naturally evaluate vector fields along Φ with the metric via the morphism $\mathcal{T}_\Phi \otimes \mathcal{T}_\Phi \rightarrow \phi_* \mathcal{O}_M$ given in coordinates (η_i) by

$$\langle \hat{\partial}_i, \hat{\partial}_j \rangle = \langle \phi^* \circ \partial_i, \phi^* \circ \partial_j \rangle := \phi^* \langle \partial_i, \partial_j \rangle. \quad (4.1)$$

4.2 Connections

Let (M, \mathcal{O}_M) be a supermanifold and \mathcal{E} a locally free sheaf of \mathcal{O}_M -super modules on M . A *connection* on \mathcal{E} (cf. [DelMor 1999]) is an even morphism $\nabla : \mathcal{E} \rightarrow \Omega_M^1 \otimes \mathcal{E}$ of sheaves of \mathbf{R} -super vector spaces that satisfies the Leibniz rule

$$\nabla(fv) = df \otimes v + f\nabla v \quad (4.2)$$

for all sections f of \mathcal{O}_M and v of \mathcal{E} . If we define

$$\nabla_X v := \langle X, \nabla v \rangle \quad (4.3)$$

for any vector field X , where $\langle X, \alpha \otimes v \rangle := \langle X, \alpha \rangle v$, we get

$$\nabla_X f v = X f \cdot v + (-1)^{|X||f|} f \nabla_X v \quad \text{and} \quad |\nabla_X v| = |X| + |v|. \quad (4.4)$$

In the case $\mathcal{E} = \mathcal{T}_M$ (in this case we speak of a connection on M) we define the *torsion* of a connection ∇ on \mathcal{T}_M by

$$T_\nabla(X, Y) := \nabla_X Y - (-1)^{|X||Y|} \nabla_Y X - [X, Y]. \quad (4.5)$$

An easy calculation shows that for any superfunction f ,

$$T_\nabla(fX, Y) = (-1)^{|f||X|} T_\nabla(X, fY) = f T_\nabla(X, Y).$$

Note that the tensorial property, namely that the values of the tensor depend only on the values of the inserted vector fields, is true in this context; nevertheless, it is less useful since a vector field cannot be reconstructed from its values.

The *curvature* of ∇ is by definition

$$R(X, Y)Z := [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z. \quad (4.6)$$

If M is furnished with a graded Riemannian metric $\langle \cdot, \cdot \rangle$, we call a connection ∇ *metric* if

$$X \langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + (-1)^{|X||Y|} \langle Y, \nabla_X Z \rangle. \quad (4.7)$$

We also have the usual notion of *Christoffel symbols*: If (η_i) is a system of coordinates (both even and odd) on $U \subset M$,

$$\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k \quad (4.8)$$

gives well-defined elements $\Gamma_{ij}^k \in \mathcal{O}_M(U)$ of parity

$$|\Gamma_{ij}^k| = |\eta_i| + |\eta_j| + |\eta_k|. \quad (4.9)$$

If it is necessary to distinguish between the even and the odd coordinates, in order not to let the notation explode the indices (latin or greek) have to indicate which Christoffel symbol is meant, e.g. $\nabla_{\partial_{\xi_\alpha}} \partial_{x_i} = \sum_j \Gamma_{\alpha i}^j \partial_{x_j} + \sum_\beta \Gamma_{\alpha i}^\beta \partial_{\xi_\beta}$.

Note that a connection on M in a natural way induces a connection on the vector bundle $TM \rightarrow M_{\text{red}}$ by reduction of the coefficient functions.

The following theorem, whose proof is – apart from the additional signs – the same as in standard (pseudo-)Riemannian geometry, can be found in [MonSan 1996]:

Theorem 4.1. *On a supermanifold M with a graded Riemannian metric, there exists a unique torsionless and metric connection ∇ (which will be called the Levi-Civita connection of the metric). It is implicitly defined by the formula*

$$\begin{aligned} 2\langle \nabla_X Y, Z \rangle &= X\langle Y, Z \rangle - (-1)^{|Z|(|X|+|Y|)} Z\langle X, Y \rangle \\ &\quad + (-1)^{|X|(|Y|+|Z|)} Y\langle Z, X \rangle + \langle [X, Y], Z \rangle \\ &\quad - (-1)^{|X|(|Y|+|Z|)} \langle [Y, Z], X \rangle \\ &\quad + (-1)^{|Z|(|X|+|Y|)} \langle [Z, X], Y \rangle. \end{aligned} \quad (4.10)$$

The Levi-Civita connection of the metric induces the standard Levi-Civita connection of the induced pseudo-Riemannian metric on M_{red} , as can be seen by regarding (4.10) on the level of the even tangent spaces $(T_p M)_0$ – there, the formula becomes the usual formula defining the Levi-Civita connection.

Consequently, if we are in coordinates (x_i, ξ_α) on U , the projections of the Christoffel symbols Γ_{ij}^k on $\mathcal{C}^\infty(U)$ are the Christoffel symbols of the Levi-Civita connection on M_{red} with respect to the corresponding coordinates on M_{red} .

Proposition 4.2. *Let M be a supermanifold with a graded Riemannian metric and the corresponding Levi-Civita connection. Then the following equalities hold for all vector fields X, Y, Z, W :*

$$\langle R(X, Y)Z, W \rangle = -(-1)^{|X||Y|} \langle R(Y, X)Z, W \rangle = -(-1)^{|Z||W|} \langle R(X, Y)W, Z \rangle \quad (4.11)$$

$$\langle R(X, Y)Z, W \rangle = (-1)^{(|X|+|Y|)(|Z|+|W|)} \langle R(Z, W)X, Y \rangle \quad (4.12)$$

$$R(X, Y)Z + (-1)^{|Z|(|X|+|Y|)} R(Z, X)Y + (-1)^{|X|(|Y|+|Z|)} R(Y, Z)X = 0. \quad (4.13)$$

Proof. Except for the additional signs, these relations are proven just like in the standard theory; as an example, we calculate one part of (4.11).

$$\begin{aligned} \langle R(X, Y)Z, W \rangle &= \langle \nabla_X \nabla_Y Z, W \rangle - (-1)^{|X||Y|} \langle \nabla_Y \nabla_X Z, W \rangle - \langle \nabla_{[X, Y]} Z, W \rangle \\ &= X\langle \nabla_Y Z, W \rangle - (-1)^{|X|(|Y|+|Z|)} \langle \nabla_Y Z, \nabla_X W \rangle \\ &\quad - (-1)^{|X||Y|} Y\langle \nabla_X Z, W \rangle + (-1)^{|Y||Z|} \langle \nabla_X Z, \nabla_Y W \rangle \\ &\quad - [X, Y]\langle Z, W \rangle + (-1)^{|Z|(|X|+|Y|)} \langle Z, \nabla_{[X, Y]} W \rangle \\ &= XY\langle Z, W \rangle - (-1)^{|Y||Z|} X\langle Z, \nabla_Y W \rangle \\ &\quad - (-1)^{|X|(|Y|+|Z|)} \langle \nabla_Y Z, \nabla_X W \rangle - (-1)^{|X||Y|} YX\langle Z, W \rangle \\ &\quad + (-1)^{|X|(|Y|+|Z|)} Y\langle Z, \nabla_X W \rangle + (-1)^{|Y||Z|} \langle \nabla_X Z, \nabla_Y W \rangle \\ &\quad - [X, Y]\langle Z, W \rangle + (-1)^{|Z|(|X|+|Y|)} \langle Z, \nabla_{[X, Y]} W \rangle \\ &= -(-1)^{|Z||W|} \langle R(X, Y)W, Z \rangle. \end{aligned}$$

Note that (4.12) is a consequence of (4.11) and (4.13) – this calculation is explicitly carried out in [Cor 2003], Lemma 1. \square

4.3 Covariant Derivatives along Supercurves

Let $\gamma = (\gamma, \gamma^*) : \mathbf{R}^{1|1} \rightarrow (M, \mathcal{O}_M)$ be a supercurve in a supermanifold with a connection ∇ . Let (t, ξ) be the standard coordinates on $\mathbf{R}^{1|1}$. Recall (2.5) that if (η_i) are coordinates on $U \subset M$ (for the moment we do not distinguish between even and odd coordinates), then

$$d\gamma(\partial_t) = \partial_t \circ \gamma^* = \sum \partial_t(\gamma^* \eta_i) \cdot \hat{\partial}_{\eta_i}.$$

We define the covariant derivative of a vector field $X = \sum f_j \hat{\partial}_{\eta_j}$ along γ with respect to the even coordinate by

$$\begin{aligned} \frac{\nabla}{dt} X &= \sum_j (\partial_t f_j) \cdot \hat{\partial}_{\eta_j} + f_j \frac{\nabla}{dt} \hat{\partial}_{\eta_j} \\ &= \sum_j (\partial_t f_j) \cdot \hat{\partial}_{\eta_j} + f_j \sum_i \partial_t(\gamma^* \eta_i) \cdot \gamma^* \nabla_{\partial_{\eta_i}} \hat{\partial}_{\eta_j} \\ &= \sum_k \left(\partial_t f_k + \sum_{i,j} f_j \cdot \partial_t(\gamma^* \eta_i) \cdot \gamma^* \Gamma_{ij}^k \right) \hat{\partial}_{\eta_k}. \end{aligned} \quad (4.14)$$

Note that the $\gamma^* \Gamma_{ij}^k$ are sections of $\mathcal{O}_{\mathbf{R}^{1|1}}$ over U , i.e. elements of $\mathcal{C}^\infty(\gamma^{-1}(U))[\xi]$; furthermore, they are homogeneous and hence either elements of $\mathcal{C}^\infty(\gamma^{-1}(U))$ or $\mathcal{C}^\infty(\gamma^{-1}(U)) \cdot \xi$, depending on the parity given by (4.9).

Analogously to (4.14), we can define the covariant derivative along a curve with respect to the odd coordinate – we only have to take into account that it has to become an odd operator:

$$\frac{\nabla}{d\xi} X = \sum_k \left(\partial_\xi f_k + \sum_{i,j} (-1)^{|f_j|} f_j \cdot \partial_\xi(\gamma^* \eta_i) \cdot \gamma^* \Gamma_{ij}^k \right) \hat{\partial}_{\eta_k}. \quad (4.15)$$

If M is now assumed to be Riemannian and equipped with a metric connection, we get the following:

Proposition 4.3. *For vector fields X and Y along a supercurve γ , we have*

$$\begin{aligned} \partial_t \langle X, Y \rangle &= \left\langle \frac{\nabla}{dt} X, Y \right\rangle + \left\langle X, \frac{\nabla}{dt} Y \right\rangle \quad \text{and} \\ \partial_\xi \langle X, Y \rangle &= \left\langle \frac{\nabla}{d\xi} X, Y \right\rangle + (-1)^{|X|} \left\langle X, \frac{\nabla}{d\xi} Y \right\rangle. \end{aligned}$$

Proof. Formula (2.5) and the metricity yield

$$\begin{aligned} \partial_t \langle \hat{\partial}_i, \hat{\partial}_j \rangle &= \partial_t \circ \gamma^* \langle \partial_i, \partial_j \rangle = \sum_k \partial_t(\gamma^* \eta_k) \cdot \gamma^* \circ \partial_k \langle \partial_i, \partial_j \rangle \\ &= \sum_k \partial_t(\gamma^* \eta_k) \cdot \gamma^* (\langle \nabla_{\partial_k} \partial_i, \partial_j \rangle + (-1)^{|\eta_i||\eta_k|} \langle \partial_i, \nabla_{\partial_k} \partial_j \rangle) \\ &= \left\langle \frac{\nabla}{dt} \hat{\partial}_i, \hat{\partial}_j \right\rangle + \left\langle \hat{\partial}_i, \frac{\nabla}{dt} \hat{\partial}_j \right\rangle; \end{aligned}$$

the second equality is proven analogously. \square

4.4 Geodesics

Before giving with (4.21) the right condition for being a supergeodesic we start with some false ones at first sight resembling the known condition from Riemannian geometry.

Let a supermanifold M be equipped with a graded Riemannian metric and the corresponding Levi-Civita connection. One generalization of the ordinary definition of geodesics would be to demand the vanishing of the term $\frac{\nabla}{dt}(d\gamma(\partial_t))$ and possibly additionally of some of the three terms $\frac{\nabla}{dt}(d\gamma(\partial_\xi))$, $\frac{\nabla}{d\xi}(d\gamma(\partial_t))$ and $\frac{\nabla}{d\xi}(d\gamma(\partial_\xi))$. Let us first regard supercurves γ that satisfy only $\frac{\nabla}{dt}(d\gamma(\partial_t)) = 0$. In local coordinates, this condition is

$$\partial_t^2(\gamma^*\eta_k) + \sum_{i,j} \partial_t(\gamma^*\eta_j) \cdot \partial_t(\gamma^*\eta_i) \cdot \gamma^*\Gamma_{ij}^k = 0 \quad (4.16)$$

for all k . It is quite useful to write these equations separately for the even and the odd coordinates. If such (x_i, ξ_α) are fixed, we have $\gamma^*x_i = g_i$ and $\gamma^*\xi_\alpha = h_\alpha\xi$ for some $g_i, h_\alpha \in \mathcal{C}^\infty(\gamma^{-1}(U))$. Then for the even coordinates, the geodesic equations are

$$\begin{aligned} 0 &= g_k'' + \sum_{i,j} g_i' g_j' \gamma^*\Gamma_{ij}^k + \sum_{i,\beta} g_i' h_\beta' \underbrace{\xi \gamma^*\Gamma_{i\beta}^k}_{=0} \\ &\quad + \sum_{\alpha,j} h_\alpha' g_j' \underbrace{\xi \gamma^*\Gamma_{\alpha j}^k}_{=0} + \sum_{\alpha,\beta} h_\alpha' h_\beta' \underbrace{\xi^2 \gamma^*\Gamma_{\alpha\beta}^k}_{=0} \\ &= g_k'' + \sum_{i,j} g_i' g_j' \gamma^*\Gamma_{ij}^k, \end{aligned} \quad (4.17)$$

where we used (4.9) for the second and third sum – there, we have Christoffel symbols with an odd number of greek indices. These are the usual geodesic equations of the underlying Riemannian manifold, which is an argument in favor of the definition: a geodesic in a supermanifold should be a supercurve with the underlying curve being an ordinary geodesic and satisfying some kind of additional odd condition.

We also write down the equations for the odd coordinates:

$$0 = h_\delta'' \xi + \sum_{i,j} g_i' g_j' \gamma^*\Gamma_{ij}^\delta + 2 \sum_{i,\beta} g_i' h_\beta' \xi \gamma^*\Gamma_{i\beta}^\delta. \quad (4.18)$$

In each of the summands appears exactly one ξ so we reduced the equations to ordinary differential equations of second order.

What are the initial conditions for geodesics of this kind in coordinate-free notation? The value of $\gamma : \mathbf{R}^{1|1} \rightarrow M$ at 0 is (in coordinates) given by the tuple $(g_i(0))$. Let us write $d_0\gamma : T_0\mathbf{R}^{1|1} \rightarrow T_{\gamma(0)}M$ in coordinates:

$$d\gamma(\partial_t|_0) = \partial_t|_0 \circ \gamma^* = \sum_i g_i'(0) \partial_{x_i}|_{\gamma(0)} \quad (4.19)$$

$$d\gamma(\partial_\xi|_0) = \partial_\xi|_0 \circ \gamma^* = \sum_\alpha h_\alpha(0) \partial_{\xi_\alpha}|_{\gamma(0)} \quad (4.20)$$

where we used that the values of nilpotent functions are zero at any point. We thus see that such a geodesic is not determined by its value and the value of its differential at one point. One might argue that this is unavoidable because of the unimportant role of points in the world of supermanifolds, but then one would have to accept that the initial conditions can not be written in a natural way in coordinate-free notation. It would thus be favourable if the odd geodesic equations (4.18) were differential equations of first order.

The additional vanishing of some of the three terms $\frac{\nabla}{dt}d\gamma(\partial_\xi)$, $\frac{\nabla}{d\xi}d\gamma(\partial_t)$ and $\frac{\nabla}{d\xi}d\gamma(\partial_\xi)$ does not solve the problem described above, whereas it is solved via the following definition - recall that the tilde means passing to the reduced level.

Definition 4.4. *A curve $\gamma : \mathbf{R}^{1|1} \rightarrow M$ is a (super)geodesic if*

$$\left(\frac{\nabla}{dt}d\gamma(\partial_t)\right)^\sim = \left(\frac{\nabla}{dt}d\gamma(\partial_\xi)\right)^\sim = 0. \quad (4.21)$$

Easy calculations in coordinates show that for all supercurves γ , we have $\frac{\nabla}{d\xi}d\gamma(\partial_\xi) = 0$ and $\left(\frac{\nabla}{dt}d\gamma(\partial_\xi)\right)^\sim = \left(\frac{\nabla}{d\xi}d\gamma(\partial_t)\right)^\sim$.

Proposition 4.5. *A curve $\gamma : \mathbf{R}^{1|1} \rightarrow M$ is a supergeodesic if and only if the underlying curve $\tilde{\gamma} : \mathbf{R} \rightarrow M$ is a geodesic and $\left(\frac{\nabla}{dt}d\gamma(\partial_\xi)\right)^\sim = 0$; in coordinates, the second condition is*

$$h'_\delta + \sum_{i,\beta} g'_i h_\beta \gamma^* \Gamma_{i\beta}^\delta = 0 \quad (4.22)$$

for all δ (the so-called odd geodesic equations).

Furthermore, for every $p \in M$ and every tangent vector $\tau \in T_p M$, there exists a unique supergeodesic γ with $\gamma(0) = p$ and $d_0\gamma(\partial_t + \partial_\xi) = \tau$.

Proof. The first statement is clear by simply writing it down in coordinates. Since the odd differential equations (4.22) are of first order, the second statement follows by (4.19) and (4.20). \square

The formal consequences of this definition are thus just what we wanted them to be; on the other hand, this notion of supergeodesic seems highly unnatural compared to the ones suggested before. Nevertheless, it will turn out that it reflects the bundle structure of a supermanifold in a very natural way.

If a supermanifold M is given, the Theorem of Batchelor allows us to find a vector bundle $\pi : E \rightarrow M_{\text{red}}$ such that M is given by the sheaf of sections of the exterior bundle $\Lambda E \rightarrow M_{\text{red}}$, cf. 2.1. The data of a supercurve $\gamma = (\tilde{\gamma}, \gamma^*) : \mathbf{R}^{1|1} \rightarrow M$ is now the same as the data of an ordinary curve $\tilde{\gamma} : \mathbf{R} \rightarrow E$: If a local basis ξ_α of E is given (which at the same time is a local system of odd coordinates of M), then $\gamma^*\xi_\alpha = g_\alpha \xi$ for some \mathcal{C}^∞ -functions g_α . Then we define a lift $\tilde{\gamma}(t) : \mathbf{R} \rightarrow E$ of $\tilde{\gamma}$ by $\tilde{\gamma}(t) = \sum_\alpha g_\alpha(t) \xi_\alpha(\tilde{\gamma}(t))$; note that the expression $\xi_\alpha(\tilde{\gamma}(t))$ makes sense since we regard the ξ_α as local sections of E . Clearly, this correspondence between supercurves in M and curves in E is one-to-one.

We want to justify Definition 4.4 by proving that the property of γ being a supergeodesic is equivalent to a natural property of the associated curve $\bar{\gamma}$. The vector bundle E carries a natural connection induced from the connection on the supermanifold M , defined as follows: Let (x_i, ξ_α) be coordinates on M such that the ξ_α are a local basis of the vector bundle E . For a vector field X on M_{red} , we define

$$\bar{\nabla}_X \xi_\alpha := (\nabla_X \partial_{\xi_\alpha})^E,$$

where the superscript E is supposed to mean the following: write the odd vector field $\nabla_X \partial_{\xi_\alpha}$ in coordinates, reduce the coefficient functions and replace ∂_{ξ_β} by ξ_β to arrive at a section of E . As an example, $(\xi_1 \partial_x + (x^2 + \xi_1 \xi_2) \partial_{\xi_2})^E = x^2 \xi_2$. Note that $\bar{\nabla}$ really is a connection on E . Now Proposition 4.5 shows that γ is a supergeodesic if and only if $\bar{\gamma}$ is the horizontal lift of a geodesic in M_{red} , where horizontality is meant with respect to $\bar{\nabla}$.

4.5 Parallel Displacement

A vector field X along a supercurve γ in a Riemannian supermanifold M , equipped with the Levi-Civita connection, is called *parallel* if

$$\left(\frac{\nabla}{dt} X \right)^\sim = \left(\frac{\nabla}{d\xi} X \right)^\sim = 0. \quad (4.23)$$

Proposition 4.6. *For each tangent vector $\tau \in T_{\gamma(0)}M$ there is a unique parallel vector field X along γ with value τ at 0. Furthermore, if τ is homogeneous, X is homogeneous of the same parity.*

Proof. Writing the conditions (4.23) for a vector field

$$X = \sum_k (f_j + \xi g_j) \hat{\partial}_j + \sum_\beta (f_\beta + \xi g_\beta) \hat{\partial}_\beta$$

along γ (where f_j, g_j, f_β and g_β are smooth functions) in coordinates (x_i, ξ_α) , we see that X is parallel if and only if

$$f'_k + \sum_{i,j} f_j \cdot (\partial_t \gamma^* x_i)^\sim \cdot (\gamma^* \Gamma_{ij}^k)^\sim = g_k + \sum_{\alpha,\beta} f_\beta \cdot (\partial_\xi \gamma^* \xi_\alpha)^\sim \cdot (\gamma^* \Gamma_{\alpha\beta}^k)^\sim = 0$$

for all k and

$$f'_\delta + \sum_{i,\beta} f_\beta \cdot (\partial_t \gamma^* x_i)^\sim \cdot (\gamma^* \Gamma_{i\beta}^\delta)^\sim = g_\delta + \sum_{\alpha,j} f_j \cdot (\partial_\xi \gamma^* \xi_\alpha)^\sim \cdot (\gamma^* \Gamma_{\alpha j}^\delta)^\sim = 0$$

for all δ . The initial condition is given by the tuples $(f_j(0))$ and $(f_\beta(0))$, so the unique existence follows. If τ is even, all the $f_\beta(0)$ are equal to 0. Thus, the f_β and g_j vanish completely and hence X is even. If τ is odd, all the $f_j(0)$ are equal to 0 and thus the f_j and g_β vanish, so X is odd. \square

We thus may define *parallel displacement* $P(\gamma)_s^t : T_{\gamma(s)}M \rightarrow T_{\gamma(t)}M$ just like in the standard theory: if $\tau \in T_{\gamma(s)}M$ is given, let X be the unique parallel vector field along γ with $X_s = \tau$ and define $P(\gamma)_s^t(\tau) := X_t$.

Proposition 4.7. *Parallel displacement is an isometry.*

Proof. For any two parallel vector fields X and Y along γ , the superfunction

$$\partial_t \langle X, Y \rangle = \left\langle \frac{\nabla}{dt} X, Y \right\rangle + \left\langle X, \frac{\nabla}{dt} Y \right\rangle$$

is nilpotent. Thus, the function $t \mapsto \langle X_t, Y_t \rangle_{\tilde{\gamma}(t)}$ is constant. \square

4.6 Isometries

Let M and N be Riemannian supermanifolds, equipped with their Levi-Civita connections. We say that a diffeomorphism $\Phi = (\phi, \phi^*) : M \rightarrow N$ is an *isometry* if it respects the metric: $\Phi^* \langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle$, i.e.

$$\phi^* \langle d\Phi(X), d\Phi(Y) \rangle = \phi^* \langle (\phi^{-1})^* \circ X \circ \phi^*, (\phi^{-1})^* \circ Y \circ \phi^* \rangle = \langle X, Y \rangle$$

for all vector fields X, Y on M . (Recall the different definitions for $d\Phi(X)$ given in 2.2 – since Φ is a diffeomorphism we regard it as a vector field on N .)

If $\Phi : M \rightarrow N$ is an isometry, Φ is in particular *affine*, i.e.

$$d\Phi(\nabla_X Y) = \nabla_{d\Phi(X)} d\Phi(Y)$$

for all vector fields X, Y on M , as can be seen by regarding the formula defining the Levi-Civita connection, (4.10). Consequently, if η_i are coordinates on U , Γ_{ij}^k the Christoffel symbols with respect to these coordinates and $\bar{\Gamma}_{ij}^k$ the Christoffel symbols with respect to the coordinates $(\phi^{-1})^* \eta_i$ on $\phi(U)$, then $\phi^* \bar{\Gamma}_{ij}^k = \Gamma_{ij}^k$.

The following lemma is clear.

Lemma 4.8. *Let $\gamma : \mathbf{R}^{1|1} \rightarrow M$ be a supergeodesic and $\Phi : M \rightarrow N$ an isometry. Then $\Phi \circ \gamma$ is a supergeodesic.*

Lemma 4.9. *Let M be a supermanifold, (x_i, ξ_α) local coordinates on $U \subset M$ and p some point of U . Let furthermore f be a function on U of degree one with respect to these coordinates. If for all supergeodesics γ starting at p we have $\gamma^* f = 0$, then $f = 0$ on some neighbourhood of p .*

Proof. Write f in these coordinates: $f = \sum_\alpha f_\alpha \xi_\alpha$ for some \mathcal{C}^∞ -functions f_α . Assume that f is not zero in any neighbourhood of p ; then we can find some q near p such that there exists an (ordinary) geodesic $\tilde{\gamma}$ with $\tilde{\gamma}(0) = p$, $\tilde{\gamma}(1) = q$ and such that not all $f_\alpha(q) = 0$. Let γ be the supergeodesic with underlying curve $\tilde{\gamma}$ that satisfies the following condition: If $\gamma^* \xi_\alpha = h_\alpha \cdot \xi$ for some smooth functions h_α , then $h_\alpha(1) = f_\alpha(q)$ (recall that the odd differential equations (4.22) are of first order). Then we have

$$(\gamma^* f)[1] = \sum_\alpha f_\alpha(\tilde{\gamma}(1))(\gamma^* \xi_\alpha)[1] = \sum_\alpha f_\alpha^2(q) \xi \neq 0,$$

which contradicts the assumption. Here, for a superfunction $g = g_0 + g_1 \xi$ on $\mathbf{R}^{1|1}$ written in coordinates, $g[1]$ shall denote the element of $\Lambda_{\mathbf{R}}[\xi]$ one gets by inserting 1 into the coefficient functions, i.e. $g[1] = g_0(1) + g_1(1)\xi$. \square

Remark. For superfunctions of degree zero, i.e. \mathcal{C}^∞ -functions, the analogous statement follows directly from the fact that the exponential map is a local diffeomorphism. For superfunctions of higher degree, the lemma is false.

Before we are able to prove that isometries are determined by the data at one point (just as geodesics, cf. Proposition 4.5), we need a technical lemma:

Lemma 4.10. *Let M be a supermanifold with odd dimension q and (x_i, ξ_α) coordinates on $U \subset M$. If f_1, \dots, f_q are superfunctions on U such that*

$$\partial_\alpha f_\beta = \partial_\beta f_\alpha$$

for all α, β , then all the f_α are the sum of functions of degree at most one (with respect to the chosen coordinates).

Proof. Assume the contrary, i.e. at least one of the functions contains some summand of degree higher than one, say f_1 . Then there are $1 \leq \alpha, \beta \leq q$ such that

$$f_1 = \xi_\alpha \xi_\beta g_1 + \xi_\alpha g_2 + \xi_\beta g_3 + g_4$$

for superfunctions g_1, \dots, g_4 not containing ξ_α and ξ_β with at least $g_1 \neq 0$. Then,

$$\partial_1 f_\alpha = \partial_\alpha f_1 = \xi_\beta g_1 + g_2, \quad (4.24)$$

and thus

$$f_\alpha = \xi_1 (\xi_\beta g_1 + g_2) + g_5$$

for some function g_5 not containing ξ_1 . Note also that we see from (4.24) that g_1 does not contain ξ_1 since both g_1 and g_2 do not contain ξ_β . Analogously,

$$\partial_1 f_\beta = \partial_\beta f_1 = -\xi_\alpha g_1 + g_3$$

implies

$$f_\beta = \xi_1 (-\xi_\alpha g_1 + g_3) + g_6$$

for some g_6 not containing ξ_1 . But then on the one hand

$$\partial_\beta f_\alpha = -\xi_1 g_1 + \partial_\beta g_5$$

and on the other hand

$$\partial_\beta f_\alpha = \partial_\alpha f_\beta = \xi_1 g_1 + \partial_\alpha g_6$$

which is a contradiction since $g_1 \neq 0$ and g_1, g_5 and g_6 do not contain ξ_1 . \square

Proposition 4.11. *An isometry of a connected Riemannian supermanifold M is determined by its value and its derivative at one point.*

Proof. Let $\Phi = (\phi, \phi^*) : M \rightarrow M$ be an isometry and $p \in M_{\text{red}}$ such that $\phi(p) = p$ and $d_p \Phi = \text{id}_{T_p M}$. We have to show $\Phi = \text{id}_M$. The corresponding result in the standard theory says that $\phi = \text{id}_{M_{\text{red}}}$.

Fix coordinates on some open set U containing p and write

$$\phi^* \xi_\alpha = \sum_\beta f_\beta^\alpha \xi_\beta + O(\xi^3)$$

for some \mathcal{C}^∞ -functions f_β^α . For every supergeodesic γ starting at p , we have $\Phi \circ \gamma = \gamma$ because of Lemma 4.8 and Proposition 4.5. In coordinates, this means

$$\gamma^* \left(\xi_\alpha - \sum_\beta f_\beta^\alpha \xi_\beta \right) = 0$$

for all α . Lemma 4.9 now says that $\sum_\beta f_\beta^\alpha \xi_\beta = \xi_\alpha$, probably after restricting U . But since the ξ_α are linearly independent as elements of the $\mathcal{C}^\infty(U)$ -module $\mathcal{O}_M(U)$, $f_\beta^\alpha = \delta_{\alpha\beta}$. Summing up, we have shown

$$\phi^* x_i = x_i + O(\xi^2), \quad \phi^* \xi_\alpha = \xi_\alpha + O(\xi^3).$$

We continue by showing that the terms of higher degree vanish, starting with the smallest: Writing

$$\phi^* x_i = x_i + f_2^i + O(\xi^4),$$

where f_2^i is homogeneous of degree 2 with respect to the \mathbf{Z} -grading, we calculate

$$d\Phi(\partial_i) = \partial_i + \sum_j (\partial_i f_2^j) \partial_j + O(\xi^3), \quad d\Phi(\partial_\alpha) = \partial_\alpha + \sum_j (\partial_\alpha f_2^j) \partial_j + O(\xi^2),$$

where the O -notation is to be understood for the coefficient functions, when the vector fields are expressed in the basis $\partial_i, \partial_\alpha$. Since Φ is an isometry,

$$\langle \partial_i, \partial_\alpha \rangle = \phi^* \langle d\Phi(\partial_i), d\Phi(\partial_\alpha) \rangle = \langle \partial_i, \partial_\alpha \rangle + \sum_j (\partial_\alpha f_2^j) \langle \partial_i, \partial_j \rangle + O(\xi^2).$$

Thus,

$$\sum_j (\partial_\alpha f_2^j) \langle \partial_i, \partial_j \rangle = 0.$$

The matrix $(\langle \partial_i, \partial_j \rangle)_{i,j}$ is invertible, so we get $\partial_\alpha f_2^i = 0$ for all α . Since f_2^i is homogeneous of degree 2, this is only possible if $f_2^i = 0$. Thus,

$$\phi^* x_i = x_i + O(\xi^4).$$

Now we deal with the odd coordinates: From

$$\phi^* \xi_\alpha = \xi_\alpha + f_3^\alpha + O(\xi^5)$$

for some homogeneous functions f_3^α of degree 3 we get

$$d\Phi(\partial_\alpha) = \partial_\alpha + \sum_\beta (\partial_\alpha f_3^\beta) \partial_\beta + O(\xi^3).$$

Then

$$\langle \partial_\alpha, \partial_\delta \rangle = \langle \partial_\alpha, \partial_\delta \rangle + \sum_\beta \left((\partial_\delta f_3^\beta) \langle \partial_\alpha, \partial_\beta \rangle - (\partial_\alpha f_3^\beta) \langle \partial_\delta, \partial_\beta \rangle \right) + O(\xi^3),$$

which implies

$$\begin{aligned} \partial_\delta \left(\sum_\beta f_3^\beta \langle \partial_\alpha, \partial_\beta \rangle^\sim \right) &= \sum_\beta (\partial_\delta f_3^\beta) \langle \partial_\alpha, \partial_\beta \rangle^\sim \\ &= \sum_\beta (\partial_\alpha f_3^\beta) \langle \partial_\delta, \partial_\beta \rangle^\sim = \partial_\alpha \left(\sum_\beta f_3^\beta \langle \partial_\delta, \partial_\beta \rangle^\sim \right). \end{aligned}$$

But now Lemma 4.10 shows that $\sum_\beta f_3^\beta \langle \partial_\alpha, \partial_\beta \rangle^\sim = 0$ for all α . Since the matrix $(\langle \partial_\alpha, \partial_\beta \rangle^\sim)_{\alpha, \beta}$ is invertible, all the $f_3^\alpha = 0$; we have shown

$$\phi^* \xi_\alpha = \xi_\alpha + O(\xi^5).$$

It is clear that we can proceed inductively on the degree the same way, alternatingly dealing with the even and odd coordinates – we only used that the f_2^i and f_3^α are homogeneous of degree greater than one. Then we see that Φ is the identity on U ; an easy argument using the connectedness of M now proves that Φ is the identity on the whole of M . \square

4.7 Graded Killing Fields

Before attacking the problem of introducing more structure to the isometry group of a Riemannian supermanifold, we deal with the corresponding infinitesimal objects: the Killing vector fields.

Let M be a Riemannian supermanifold. A *graded Killing vector field* on M is a vector field X such that

$$X \langle Y, Z \rangle = \langle [X, Y], Z \rangle + (-1)^{|X||Y|} \langle Y, [X, Z] \rangle \quad (4.25)$$

for all Y, Z . Thus, using the properties of the Levi-Civita connection ∇ , X is a graded Killing vector field if and only if

$$\langle \nabla_Y X, Z \rangle + (-1)^{|X||Y|+|X||Z|+|Y||Z|} \langle \nabla_Z X, Y \rangle = 0. \quad (4.26)$$

The super vector space of all graded Killing vector fields becomes a Lie superalgebra, if we define the bracket to be induced from the Lie superalgebra of all vector fields. Following usual conventions, we will denote by *the Lie superalgebra of graded Killing vector fields* the opposite to this Lie superalgebra, i.e. the Lie superalgebra with the same underlying vector space and the new bracket the negative of the old bracket.

We denote the second covariant derivative by $\nabla_{Y,Z}^2 X := \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$. Then we have

$$R(Y, Z)X = \nabla_{Y,Z}^2 X - (-1)^{|Y||Z|} \nabla_{Z,Y}^2 X. \quad (4.27)$$

Lemma 4.12. *Let X be a Killing vector field. Then*

$$\langle \nabla_{Y,Z}^2 X, W \rangle + (-1)^{|Z||X|+|Z||W|+|X||W|} \langle \nabla_{Y,W}^2 X, Z \rangle = 0.$$

Proof. This is a direct calculation using (4.26) several times. \square

Proposition 4.13. *Let X be a Killing vector field. Then*

$$\nabla_{Y,Z}^2 X = -(-1)^{|X|(|Y|+|Z|)} R(X, Y)Z.$$

Proof. The proof is analogously to [Pet 1998], p.216, with a lot of additional signs; it uses Lemma 4.12, formula (4.27) and the symmetries of the curvature tensor collected in Proposition 4.2. Since no new idea enters, we again omit the calculation. \square

Proposition 4.14. *Let X be a Killing vector field on a Riemannian supermanifold. If there exists a point p such that $X(p) = 0$ and $(\nabla X)(p) = 0$, then $X = 0$.*

Remark. In standard Riemannian geometry, this is proven either via the flow of X or the fact that X , restricted to any geodesic, is a Jacobi field. Since we did not introduce Jacobi fields and flows of odd vector fields, we need a different proof.

Proof. Consider first the case of X even. Pick coordinates (x_i, ξ_α) around p and write $X = \sum_i (\sum_j f_j^i) \partial_i$ in coordinates; we assume that the degree of f_j^i is j . Proposition 4.13, written in coordinates, is a system of second order differential equations; it yields the vanishing of the coefficient functions of degree 0 and 1, i.e.

$$X = \sum_i f_2^i \partial_i + O(\xi^3).$$

(Recall that the O -notation is to be understood for the coefficient functions.) Then on the one hand $X \langle \partial_j, \partial_\beta \rangle = O(\xi^3)$ and on the other hand

$$X \langle \partial_j, \partial_\beta \rangle = \langle [X, \partial_j], \partial_\beta \rangle + \langle \partial_j, [X, \partial_\beta] \rangle = - \sum_i (\partial_\beta f_2^i) \langle \partial_j, \partial_i \rangle + O(\xi^3).$$

Since the matrix $(\langle \partial_i, \partial_j \rangle)$ is invertible, we conclude $\partial_\beta f_2^i = 0$ for all i and β . Thus, $f_2^i = 0$, since the function was assumed to be homogeneous of positive degree.

Attacking the remaining term of smallest degree we write

$$X = \sum_\alpha f_3^\alpha \partial_\alpha + O(\xi^4).$$

Then on the one hand $X \langle \partial_\beta, \partial_\delta \rangle = O(\xi^4)$ and on the other hand

$$\begin{aligned} X \langle \partial_\beta, \partial_\delta \rangle &= \langle [X, \partial_\beta], \partial_\delta \rangle + \langle \partial_\beta, [X, \partial_\delta] \rangle \\ &= - \sum_\alpha (\partial_\beta f_3^\alpha) \langle \partial_\alpha, \partial_\delta \rangle - \sum_\alpha (\partial_\delta f_3^\alpha) \langle \partial_\beta, \partial_\alpha \rangle + O(\xi^4). \end{aligned}$$

Lemma 4.10 yields the vanishing of the f_3^α via the same argument as in the last part of the proof of Proposition 4.11; then induction finishes the case of X even.

If X is odd, we can argue very similarly, with only some signs changing. First of all, the same argument shows the vanishing of the coefficient functions of degree 0 and 1. Applying $X = \sum_{\alpha} f_2^{\alpha} \partial_{\alpha} + O(\xi^3)$ to the superfunctions $\langle \partial_{\beta}, \partial_{\delta} \rangle$ yields the vanishing of the f_2^{α} ; thereafter applying $X = \sum_i f_3^i \partial_i + O(\xi^4)$ to $\langle \partial_j, \partial_{\beta} \rangle$ shows that $f_3^i = 0$. As usual we continue by induction. \square

4.8 The Isometry Group

In this section we want to define the isometry group $I(M)$ of a Riemannian supermanifold M ; naturally, it shall become not a Lie group but a Lie supergroup. Staying consistent with the ungraded case, the Lie superalgebra of $I(M)$ has to be the Lie superalgebra of graded Killing vector fields.

The set of all isometries (cf. 4.6) of a Riemannian supermanifold M clearly is a group. Our task now is to show that there is a natural Lie structure on it. Then we will equip this Lie group with the Lie superalgebra of graded Killing vector fields to get a Harish-Chandra pair – this will by definition give us the isometry supergroup of M .

In spirit of this, we denote by $I(M)_{\text{red}}$ the group of isometries of M , although we have not yet defined $I(M)$. This is not to be mixed up with $I(M_{\text{red}})$, which is simply the usual isometry group of the pseudo-Riemannian manifold M_{red} . Recall that the isometry group of a conventional pseudo-Riemannian manifold, endowed with the compact-open topology, is a Lie group, see e.g. Ballmann [Bal 2000]: it is a closed subgroup of the group of affine diffeomorphisms with respect to the Levi-Civita connection of the metric.

It would be nice to say that we equip $I(M)_{\text{red}}$ with the compact-open topology; but since topological notions like compactness and openness make sense only for the underlying manifold, this is not possible. In the following we therefore want to realize $I(M)_{\text{red}}$ as a closed subgroup of an automorphism group of the parallelization of some ordinary differentiable manifold and then use the general result of Ballmann, namely that such groups are Lie groups.

Let us briefly recall the definitions of [Bal 2000]: If N is a conventional n -dimensional manifold together with a parallelization $\Phi : N \times \mathbf{R}^n \rightarrow TN$, any $z \in \mathbf{R}^n$ induces a so-called *constant vector field* $Z(p) = \Phi(p, z)$. The *automorphism group* $\text{Aut}(\Phi)$ of Φ then is the group of all diffeomorphisms $f : N \rightarrow N$ such that $df \circ Z = Z \circ f$ for all constant vector fields Z , equipped with the compact-open topology.

The following is an extension of Example 3.1 of [Bal 2000]: Let M be an $n|2m$ -dimensional Riemannian supermanifold with Levi-Civita connection ∇ and consider the vector bundle $TM = (TM)_0 \oplus (TM)_1 \rightarrow M_{\text{red}}$ with $n + 2m$ -dimensional fibres $T_p M$ (we may regard it as a bundle of $n|2m$ -dimensional super vector spaces). Recall that the connection on M induces a connection on $TM \rightarrow M_{\text{red}}$. Let $\pi : \text{GL}(TM) \rightarrow M_{\text{red}}$ be the $\text{GL}(n) \times \text{GL}(2m)$ -principal fibre bundle of graded frames; the fibre over $p \in M_{\text{red}}$ is given by

$$\text{GL}(TM)_p = \text{GL}(\mathbf{R}^n, (T_p M)_0) \times \text{GL}(\mathbf{R}^{2m}, (T_p M)_1).$$

We want to find a parallelization of the conventional manifold $\text{GL}(TM)$ such that affine diffeomorphisms of the supermanifold M induce diffeomorphisms of

$\mathrm{GL}(TM)$ respecting the parallelization.

The vertical distribution \mathcal{V} of $\mathrm{GL}(TM)$ given by the kernel of $d\pi$ is trivialized via the mapping

$$\Phi_v : \mathrm{GL}(TM) \times (\mathfrak{gl}(n) \times \mathfrak{gl}(2m)) \rightarrow \mathcal{V}$$

defined by

$$\Phi_v((\phi_1, \phi_2), (x_1, x_2)) = \partial_t (\phi_1 \exp(tx_1), \phi_2 \exp(tx_2))|_{t=0}.$$

It remains to find a corresponding (trivializable) horizontal distribution \mathcal{H} (which will be of rank n). For any $\phi = (\phi_1, \phi_2) \in \mathrm{GL}(TM)$ and $z \in \mathbf{R}^n$, let $v = \phi_1(z) \in T_p M_{\mathrm{red}}$, where $p = \pi(\phi) \in M_{\mathrm{red}}$. Choose a curve c in M_{red} through p with $c'(0) = v$ and let ψ be the unique parallel frame along c such that $\psi(0) = \phi$. Then

$$\Phi_h(\phi, z) := \psi'(0)$$

gives a trivialization

$$\Phi_h : \mathrm{GL}(TM) \times \mathbf{R}^n \rightarrow \mathcal{H}$$

of a horizontal distribution \mathcal{H} . The pair $\Phi = (\Phi_v, \Phi_h)$ therefore is a parallelization of $\mathrm{GL}(TM)$. If f is a diffeomorphism of the supermanifold M , it induces a diffeomorphism f_* of $\mathrm{GL}(TM)$ sending a frame ϕ to $df \circ \phi$; if f is affine, this diffeomorphism is an element of $\mathrm{Aut}(\Phi)$.

We thus get a mapping $I(M)_{\mathrm{red}} \rightarrow \mathrm{Aut}(\Phi)$, which is injective by Proposition 4.11.

Proposition 4.15. *The image of the natural inclusion $I(M)_{\mathrm{red}} \rightarrow \mathrm{Aut}(\Phi)$ is closed in $\mathrm{Aut}(\Phi)$.*

Proof. Let $g_n : M \rightarrow M$ be a sequence of isometries of the supermanifold M such that the induced mappings g_{n*} converge against some h with respect to the compact-open topology.

The space $\mathcal{O}_M(M)$ is a Fréchet space via the family of semi-norms $|\cdot|_{K,\partial}$ defined as follows: for any compact subset $K \subset M_{\mathrm{red}}$ and any differential operator ∂ , we set $|f|_{K,\partial} := \sup_{p \in K} |(\partial f)(p)|$, see Kostant [Kost 1975], p.199. By writing the condition of being an isometry in coordinates, an isometry can be viewed as a solution of some system of ordinary differential equations. Since the g_{n*} converge, also these solutions converge in the sense that for any $f \in \mathcal{O}_M(M)$, the sequence $g_n^* f$ converges against some $\Gamma(f) \in \mathcal{O}_M(M)$. Then, the mapping $\Gamma : \mathcal{O}_M(M) \rightarrow \mathcal{O}_M(M)$ defines an isometry g of M such that the induced mapping $g_* : \mathrm{GL}(TM) \rightarrow \mathrm{GL}(TM)$ equals h . \square

Thus, by giving $I(M)_{\mathrm{red}}$ the induced topology of $\mathrm{Aut}(\Phi)$, the results of [Bal 2000] introduce the structure of Lie group on $I(M)_{\mathrm{red}}$.

Remark. Another possible way of turning $I(M)_{\mathrm{red}}$ into a Lie group is the following: Give it the coarsest topology such that for all $f \in \mathcal{O}_M(M)$, the mapping $I(M)_{\mathrm{red}} \rightarrow \mathcal{O}_M(M); g \mapsto g^* f$ is continuous – here, $\mathcal{O}_M(M)$ carries the structure of a Fréchet space mentioned in the proof above. Then, $I(M)_{\mathrm{red}}$ is a locally compact topological group without small subgroups. It is known that such groups are Lie groups, see [MonZip 1965], p.107 and the references given there.

To complete the construction of the isometry supergroup we have to equip this Lie group with a suitable Lie superalgebra.

Let us first calculate the Lie algebra of $I(M)_{\text{red}}$. If we take a left invariant vector field X on $I(M)_{\text{red}}$, it defines a unique one-parameter group (g_t) in $I(M)_{\text{red}}$. Then the associated vector field $f \mapsto \partial_t|_{t=0}(g_t^* f)$ is an even Killing field; this correspondence is an isomorphism from the Lie algebra of $I(M)_{\text{red}}$ to the Lie algebra of even Killing fields on M .

The *isometry group* $I(M)$ of M is now the Lie supergroup associated to the Harish-Chandra pair

$$(I(M)_{\text{red}}, \mathfrak{g}),$$

where \mathfrak{g} is the Lie superalgebra of all graded Killing vector fields on M . We have already verified that the Lie algebra of $I(M)_{\text{red}}$ is equal to \mathfrak{g}_0 , so what is left is to specify the action of $I(M)_{\text{red}}$ on \mathfrak{g} . For a graded Killing vector field X on M and an isometry $g \in I(M)_{\text{red}}$, we define

$$\text{Ad}_g X := dg(X) = (g^{-1})^* \circ X \circ g^*.$$

On \mathfrak{g}_0 , this coincides with the action of $I(M)_{\text{red}}$ on its Lie algebra. Note that if M is a usual manifold, $I(M)$ is the usual isometry group.

There is a natural action of $I(M)$ on M which we give in terms of the above Harish-Chandra pair, cf. 3.3: The Lie group $I(M)_{\text{red}}$ clearly acts on M ; in fact, it is defined as a group of diffeomorphisms of M . We have to define a compatible morphism from the Lie superalgebra of graded Killing vector fields on M to the opposite of the Lie superalgebra of vector fields on M . Here, we simply take the inclusion.

4.9 Invariant Metrics on Lie Supergroups

Given an ordinary Lie group G , a pseudo-Riemannian metric $\langle \cdot, \cdot \rangle$ on G is called *left-invariant* if the left translations are isometries. Equivalent to this is the condition that for all left-invariant vector fields X and Y , $\langle X, Y \rangle$ is a constant function on G (i.e. a real multiple of the unit in the algebra of global functions on G). We take this to be the definition of left-invariance in the case of Lie supergroups. Analogously, we define right-invariance and then bi-invariance.

The left-invariant graded Riemannian metrics on a Lie supergroup G are in one-to-one correspondence with the scalar superproducts on \mathfrak{g} : for $X, Y \in \mathfrak{g}$ regard the real number $\langle X, Y \rangle_e$ as a constant function on G and extend the metric linearly with respect to superfunctions.

A scalar superproduct $\langle \cdot, \cdot \rangle_e$ on \mathfrak{g} is called $\text{Ad}_{G_{\text{red}}}$ -invariant if

$$\langle \text{Ad}_g X, \text{Ad}_g Y \rangle_e = \langle X, Y \rangle_e$$

for all $X, Y \in \mathfrak{g}$ and all $g \in G_{\text{red}}$. It is $\text{ad}_{\mathfrak{g}}$ -invariant if

$$\langle [X, Y], Z \rangle_e + (-1)^{|X||Y|} \langle Y, [X, Z] \rangle_e = 0$$

for all $X, Y, Z \in \mathfrak{g}$. Of course, if G is connected, $\text{ad}_{\mathfrak{g}}$ -invariance implies $\text{Ad}_{G_{\text{red}}}$ -invariance. We call the scalar superproduct Ad_G -invariant or simply Ad -invariant if it is $\text{Ad}_{G_{\text{red}}}$ - and $\text{ad}_{\mathfrak{g}}$ -invariant.

As an example, $G = \mathbf{R}^{1|2}$ with the Lie supergroup structure given in symbolic notation by

$$(x, \xi_1, \xi_2) + (t, \theta_1, \theta_2) := (x + t + \xi_1 \xi_2 + \theta_1 \theta_2, \xi_1 + \theta_1, \xi_2 + \theta_2)$$

does not admit any bi-invariant metric; its Lie algebra admits $\text{Ad}_{G_{\text{red}}}$ -invariant but no $\text{ad}_{\mathfrak{g}}$ -invariant scalar superproducts.

The equivalence of the bi-invariance of a metric and the ad-invariance of the induced supersymmetric bilinear form at e , which is very easily proven in the standard case, requires some preparation before we can establish it in Theorem 4.19.

Lemma 4.16. *Let G be a Lie supergroup. Then for any left-invariant vector field X and any right-invariant vector field Y , we have $[X, Y] = 0$.*

Proof. Let $\tau, \sigma \in T_e G$ and X_τ, X_σ be the corresponding left-invariant vector fields. The differential of the inverse map i interchanges left- and right-invariant vector fields, so it suffices to show $[X_\tau, di(X_\sigma)] = 0$; this is an easy calculation using (3.3). \square

Proposition 4.17. *Let G be a Lie supergroup, equipped with a left-invariant graded Riemannian metric such that the induced scalar superproduct on \mathfrak{g} is $\text{ad}_{\mathfrak{g}}$ -invariant. Then*

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \text{and} \quad R(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \quad (4.28)$$

for all left-invariant vector fields X, Y .

Remark. Note that we do not yet prove the statement for bi-invariant metrics!

Proof. This is – again apart from the additional signs – the standard proof via (4.10). \square

Lemma 4.18. *Let G be a Lie supergroup with a left-invariant graded Riemannian metric $\langle \cdot, \cdot \rangle$ such that the induced scalar superproduct on \mathfrak{g} is $\text{ad}_{\mathfrak{g}}$ -invariant. Then*

$$\tau \langle X, Y \rangle = 0$$

for all right-invariant vector fields X, Y and all tangent vectors $\tau \in T_e G$.

Proof. Let τ, σ and $\rho \in T_e G$ and denote by X_τ, X_σ, X_ρ and Y_τ, Y_σ, Y_ρ the corresponding left- and right-invariant vector fields. Then we have

$$\begin{aligned} \tau \langle Y_\sigma, Y_\rho \rangle &= \langle \nabla_{X_\tau} Y_\sigma, Y_\rho \rangle(e) + (-1)^{|\tau||\sigma|} \langle Y_\sigma, \nabla_{X_\tau} Y_\rho \rangle(e) \\ &= \left\langle (-1)^{|\tau||\sigma|} \nabla_{Y_\sigma} X_\tau + [X_\tau, Y_\sigma], X_\rho \right\rangle(e) \\ &\quad + (-1)^{|\tau||\sigma|} \left\langle X_\sigma, (-1)^{|\tau||\rho|} \nabla_{Y_\rho} X_\tau + [X_\tau, Y_\rho] \right\rangle(e) \\ &= (-1)^{|\tau||\sigma|} \langle \nabla_{X_\sigma} X_\tau, X_\rho \rangle(e) + (-1)^{|\tau|(|\sigma|+|\rho|)} \langle X_\sigma, \nabla_{X_\rho} X_\tau \rangle(e) \\ &= -\frac{1}{2} \left(\langle [X_\tau, X_\sigma], X_\rho \rangle(e) + (-1)^{|\tau||\sigma|} \langle X_\sigma, [X_\tau, X_\rho] \rangle(e) \right) = 0. \end{aligned}$$

In the first and second line we used the properties of the Levi-Civita connection. Then we applied Lemma 4.16 and used the fact that for vector fields Z and W , the value of $\nabla_Z W$ at some point depends only on the value of Z at that point. In the last line we finally used Proposition 4.17 and the $\text{ad}_{\mathfrak{g}}$ -invariance of the induced scalar superproduct. \square

Theorem 4.19. *Let G be a Lie supergroup with a left-invariant graded Riemannian metric $\langle \cdot, \cdot \rangle$. Then the metric is bi-invariant if and only if $\langle \cdot, \cdot \rangle_e$ is Ad_G -invariant.*

Proof. If the metric is left-invariant (resp. right-invariant), left (resp. right) translations with elements of G_{red} are isometries of M ; thus, if $\langle \cdot, \cdot \rangle$ is bi-invariant, $\langle \cdot, \cdot \rangle_e$ is $\text{Ad}_{G_{\text{red}}}$ -invariant. To prove the $\text{ad}_{\mathfrak{g}}$ -invariance, calculate for $\tau, \sigma, \rho \in T_e G$ as follows (X_τ, X_σ, X_ρ and Y_τ, Y_σ, Y_ρ are again the corresponding left- and right-invariant vector fields):

$$\begin{aligned}
\langle [X_\tau, X_\sigma], X_\rho \rangle_e &= \left\langle \nabla_{X_\tau} X_\sigma - (-1)^{|\tau||\sigma|} \nabla_{X_\sigma} X_\tau, X_\rho \right\rangle_e \\
&= \left(X_\tau \langle X_\sigma, X_\rho \rangle(e) - (-1)^{|\tau||\sigma|} \langle X_\sigma, \nabla_{X_\tau} X_\rho \rangle_e \right) - (-1)^{|\tau||\sigma|} \langle \nabla_{Y_\sigma} X_\tau, X_\rho \rangle(e) \\
&= -(-1)^{|\tau||\sigma|} \langle X_\sigma, \nabla_{X_\tau} X_\rho \rangle_e - \left\langle \nabla_{X_\tau} Y_\sigma + (-1)^{|\tau||\sigma|} [Y_\sigma, X_\tau], Y_\rho \right\rangle(e) \\
&= -(-1)^{|\tau||\sigma|} \langle X_\sigma, \nabla_{X_\tau} X_\rho \rangle_e - \left(X_\tau \langle Y_\sigma, Y_\rho \rangle(e) - (-1)^{|\tau||\sigma|} \langle Y_\sigma, \nabla_{X_\tau} Y_\rho \rangle(e) \right) \\
&= -(-1)^{|\tau||\sigma|} \langle X_\sigma, \nabla_{X_\tau} X_\rho \rangle_e + (-1)^{|\tau|(|\sigma|+|\rho|)} \langle X_\sigma, \nabla_{Y_\rho} X_\tau \rangle(e) \\
&= -(-1)^{|\tau||\sigma|} \left\langle X_\sigma, \nabla_{X_\tau} X_\rho - (-1)^{|\tau||\rho|} \nabla_{X_\rho} X_\tau \right\rangle_e \\
&= -(-1)^{|\tau||\sigma|} \langle X_\sigma, [X_\tau, X_\rho] \rangle_e.
\end{aligned}$$

Note that we used Lemma 4.16 for the fourth and the right-invariance of $\langle \cdot, \cdot \rangle$ for the fifth equality.

Let now $\langle \cdot, \cdot \rangle_e$ be Ad_G -invariant. We want to show the bi-invariance of $\langle \cdot, \cdot \rangle$. The left-invariance of $\langle \cdot, \cdot \rangle$, together with the $\text{Ad}_{G_{\text{red}}}$ -invariance of $\langle \cdot, \cdot \rangle_e$ shows that for all right-invariant vector fields X and Y , the \mathcal{C}^∞ -part of the superfunction $\langle X, Y \rangle$ is constant. We have to show that the superfunction itself is constant.

Fix a basis $\{Z_\eta\} = \{Z_i, Z_\alpha\}$ consisting of homogeneous right-invariant vector fields; assume that the value of Z_η at e is ∂_η . We write

$$Z_\eta = \sum_i z_\eta^i \partial_i + \sum_\alpha z_\eta^\alpha \partial_\alpha$$

for superfunctions z_η^i and z_η^α . First of all, let us have a look at the case that X is even and Y is odd. We write

$$\langle X, Y \rangle = \sum_\alpha f_\alpha \xi_\alpha + O(\xi^3)$$

for some \mathcal{C}^∞ -functions f_α . Our first goal is to show that $f_\alpha = 0$. On the one hand

$$Z_\beta \langle X, Y \rangle = \sum_{\alpha, i} z_\beta^i (\partial_i f_\alpha) \xi_\alpha + \sum_\alpha z_\beta^\alpha f_\alpha + O(\xi^2) = \sum_\alpha \widetilde{z}_\beta^\alpha f_\alpha + O(\xi^2).$$

On the other hand,

$$Z_\beta \langle X, Y \rangle = \langle \nabla_{Z_\beta} X, Y \rangle + \langle X, \nabla_{Z_\beta} Y \rangle$$

is the sum of a constant function and nilpotent terms. Consequently,

$$\sum_\alpha \widetilde{z}_\beta^\alpha f_\alpha \equiv Z_\beta \langle X, Y \rangle (e) = 0$$

according to Lemma 4.18. Since the matrix $(\widetilde{z}_\beta^\alpha)_{\beta,\alpha}$ is invertible (as $\{Z_\eta\}$ is a basis) we conclude that all the f_α must vanish.

Let us now have a look at the case when X and Y are both even or both odd. Then

$$\langle X, Y \rangle = \text{const} + f_2 + O(\xi^4)$$

for some homogeneous function f_2 of degree 2 (in the chosen coordinates). Then for any γ ,

$$Z_\gamma \langle X, Y \rangle = \sum_\alpha \widetilde{z}_\gamma^\alpha (\partial_\alpha f_2) + O(\xi^3).$$

But because

$$Z_\gamma \langle X, Y \rangle = \langle \nabla_{Z_\gamma} X, Y \rangle + (-1)^{|X|} \langle X, \nabla_{Z_\gamma} Y \rangle$$

consists only of terms of degree at least three by what we have shown before, and because the matrix $(\widetilde{z}_\gamma^\alpha)$ is invertible, we see that $\partial_\alpha f_2 = 0$ for any α . The superfunction f_2 is homogeneous of degree 2, so $f_2 = 0$.

We proceed by induction: Consider the case of X even and Y odd, write

$$\langle X, Y \rangle = f_{2k+1} + O(\xi^{2k+3})$$

for some homogeneous superfunction f_{2k+1} of degree $2k+1$, apply all the Z_β and conclude that $f_{2k+1} = 0$. The analogous argument as above for the case of X and Y of the same parity finishes the induction. \square

Corollary 4.20. *Let G be a Lie supergroup with a bi-invariant graded Riemannian metric. Then*

$$\nabla_X Y = \frac{1}{2}[X, Y] \quad \text{and} \quad R(X, Y)Z = -\frac{1}{4}[[X, Y], Z] \quad (4.29)$$

for all left-invariant vector fields X, Y .

As the last result in this section, we state the following easy generalization of Proposition 1.6 b) of [CahPar 1980] – it will reduce the amount of calculation needed for the last example in 4.14. Up to signs, the proof is the same as there. This result also appears as Theorem 4 in [Cor 2003]; note that there the assumption of the adjoint representation being faithful is missing.

Proposition 4.21. *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra such that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$ and the adjoint representation of \mathfrak{g}_0 on \mathfrak{g}_1 is faithful. Then any non-degenerate $\text{ad}_{\mathfrak{g}_0}$ -invariant skew-symmetric bilinear form $\langle \cdot, \cdot \rangle_1$ on \mathfrak{g}_1 uniquely extends to an $\text{ad}_{\mathfrak{g}}$ -invariant scalar superproduct $\langle \cdot, \cdot \rangle$ on \mathfrak{g} .*

4.10 The Killing Form

Let \mathfrak{g} be a finite-dimensional Lie superalgebra. We define the *Killing form* of \mathfrak{g} to be

$$B(X, Y) := \text{str}(\text{ad}_X \circ \text{ad}_Y), \quad (4.30)$$

where str is the supertrace. The Killing form is an even graded-symmetric $\text{ad}_{\mathfrak{g}}$ -invariant bilinear form; the $\text{ad}_{\mathfrak{g}}$ -invariance meaning

$$B([X, Y], Z) + (-1)^{|X||Y|} B(Y, [X, Z]) = 0 \quad (4.31)$$

for all $X, Y, Z \in \mathfrak{g}$.

It is known (see e.g. [Sche 1979]), that on $\mathfrak{sl}(n|m)$, the Killing form is given by

$$B(X, Y) = 2(n - m) \text{str}(XY);$$

it is non-degenerate for $n \neq m$ but identically zero for $n = m$. Nevertheless, the supertrace $(X, Y) \mapsto \text{str}(XY)$ induces a non-degenerate even graded-symmetric $\text{ad}_{\mathfrak{sl}(n|m)}$ -invariant bilinear form on $\mathfrak{sl}(n|m)/K \cdot I_{2n}$, where I_{2n} is the $2n \times 2n$ unit matrix.

On $\mathfrak{osp}(n|2m)$, the Killing form is

$$B(X, Y) = (n - 2m - 2) \text{str}(XY);$$

it is non-degenerate if $n \neq 2m + 2$ and identically zero for $n = 2m + 2$. Here, the supertrace is non-degenerate and $\text{ad}_{\mathfrak{osp}(n|2m)}$ -invariant for all n and m .

Note that there exist Lie superalgebras with degenerate (even vanishing) Killing form that nevertheless admit an ad-invariant scalar superproduct.

4.11 Homogeneous Superspaces

If G is a Lie supergroup and H a closed Lie subsupergroup (a Lie subsupergroup such that G_{red} is closed in H_{red}), we know from the standard theory that $G_{\text{red}}/H_{\text{red}}$ is a manifold. Consider the canonical projections $\pi : G_{\text{red}} \rightarrow G_{\text{red}}/H_{\text{red}}$ and $\text{pr}_1 : G \times H \rightarrow G$ and the right action of H on G , $\Phi = (\phi, \phi^*) : G \times H \rightarrow G$, i.e. the composition of the multiplication morphism with the inclusion of H into G . We equip $G_{\text{red}}/H_{\text{red}}$ with the sheaf of H -invariant superfunctions:

$$\mathcal{O}_{G/H}(U) := \{f \in \mathcal{O}_G(\pi^{-1}(U)) \mid \phi^* f = \text{pr}_1^* f\}. \quad (4.32)$$

Then

$$G/H := (G_{\text{red}}/H_{\text{red}}, \mathcal{O}_{G/H})$$

is a supermanifold, see [Kost 1975], Theorem 3.9. There is a canonical morphism of supermanifolds $G \rightarrow G/H$, where the sheaf morphism is given by inclusion. The surjective morphism $\mathfrak{g} \cong T_e G \rightarrow T_H G/H$ has kernel \mathfrak{h} and thus yields a canonical identification

$$T_H G/H \cong \mathfrak{g}/\mathfrak{h}.$$

The adjoint representation of G restricts to a well-defined representation Ad_H of H on $\mathfrak{g}/\mathfrak{h}$, i.e. compatible representations

$$\text{Ad}_{H_{\text{red}}} : H_{\text{red}} \times \mathfrak{g}/\mathfrak{h} \text{ and } \text{ad}_{\mathfrak{h}} : \mathfrak{h} \times \mathfrak{g}/\mathfrak{h} \rightarrow \mathfrak{g}/\mathfrak{h}.$$

If $\rho : G \times M \rightarrow M$ is an action of a Lie supergroup G on a supermanifold M and $p \in M_{\text{red}}$, we define the *isotropy group* G_p to be the Lie subsupergroup of G given by the Harish-Chandra pair

$$((G_{\text{red}})_p, \mathfrak{g}_p),$$

where

$$\mathfrak{g}_p := \{X \in \mathfrak{g} \mid \text{ev}_p \circ (X_e \otimes I) \circ \rho^* = 0\}. \quad (4.33)$$

Here, $\text{ev}_p : \mathcal{T}_M(M) \rightarrow T_p M$ is evaluation at p ; cf. also 3.3. In other words, \mathfrak{g}_p consists of those left-invariant vector fields X on G such that the infinitesimal action of X at p is trivial.

As usual we have the *isotropy representation* of G_p on $T_p M$: On the level of the underlying Lie group, $g \in (G_{\text{red}})_p$ acts on $T_p M$ in an obvious manner; the Lie superalgebra \mathfrak{g}_p acts on $T_p M$ via

$$X \cdot v = -[(X_e \otimes I) \circ \rho^*, v]. \quad (4.34)$$

Note that this is a well-defined action compatible with the $(G_{\text{red}})_p$ -action, so they fit together to a representation of the Lie supergroup G_p .

Let $\rho : G \times M \rightarrow M$ be an action on a supermanifold M and fix $p \in M_{\text{red}}$. We can define the *orbit* $G \cdot p$ as follows: The usual orbit $G_{\text{red}} \cdot p \subset M_{\text{red}}$ inherits its differentiable structure via the canonical mapping $j : G_{\text{red}}/(G_{\text{red}})_p \rightarrow G_{\text{red}} \cdot p$. The supermanifold $G \cdot p$ is now defined to be

$$G \cdot p := (G_{\text{red}} \cdot p, j_* \mathcal{O}_{G/G_p});$$

it is no surprise that also the superstructure has to be passed over from G/G_p . If we denote the inclusion $G_{\text{red}} \cdot p \rightarrow M_{\text{red}}$ by i , the orbit map

$$\rho_p : G \xrightarrow{\text{id} \times p} G \times M \xrightarrow{\rho} M$$

of the point p gives a well-defined sheaf morphism

$$\mathcal{O}_M \rightarrow i_* \mathcal{O}_{G \cdot p} = (i \circ j)_* \mathcal{O}_{G/G_p}.$$

Note that the orbits $G \cdot p$ become submanifolds of M via this inclusion morphism $i = (i, \rho_p^*) : G \cdot p \rightarrow M$.

An action $\rho : G \times M \rightarrow M$ is said to be *transitive* (cf. [Oni 1994], p. 295) if the reduced action $\tilde{\rho} : G_{\text{red}} \times M_{\text{red}} \rightarrow M_{\text{red}}$ is transitive and if for all $p \in M_{\text{red}}$, the mapping

$$\mathfrak{g} \rightarrow T_p M; \quad X \mapsto \text{ev}_p \circ (X_e \otimes I) \circ \rho^* = X_e \circ \rho_p^* \quad (4.35)$$

is surjective. It is known from the standard theory that in the case of a connected supermanifold, the first condition is equivalent to the even part of the

second condition, i.e. the surjectivity of the mapping $\mathfrak{g}_0 \rightarrow (T_p M)_0$. We say that a supermanifold is *G-homogeneous* if it is acted on transitively by G ; we say it is *homogeneous* if it is G -homogeneous for some Lie supergroup G . If M is already assumed to be Riemannian, it is *homogeneous* if it is $I(M)$ -homogeneous.

Note that if M is G -homogeneous, the natural morphism $G \cdot p \rightarrow M$ is an isomorphism.

Proposition 4.22. *Let M be G -homogeneous. Then the isotropy representation of G_p on $T_p M$ is equivalent to the adjoint representation on $\mathfrak{g}/\mathfrak{g}_p$ via the natural isomorphism $\mathfrak{g}/\mathfrak{g}_p \rightarrow T_p M$.*

Proof. First we have a look at the representations of the underlying Lie group. For any $g \in (G_{\text{red}})_p$ and any $X \in \mathfrak{g}$, we have

$$(\text{Ad}_g X)_e \circ \rho_p^* = X_e \circ l_g^* \circ r_{g^{-1}}^* \circ \rho_p^* = X_e \circ l_g^* \circ \rho_p^* = X_e \circ \rho_p^* \circ g^* = dg(X_e \circ \rho_p^*),$$

where l_g and $r_g : G \rightarrow G$ are left and right multiplication with g , and the diffeomorphism of M induced by the action of g is again denoted by $g : M \rightarrow M$.

The equivalence of the representations at the level of Lie superalgebras is the equation

$$[Y, X]_e \circ \rho_p^* = -[(Y_e \otimes I) \circ \rho^*, X_e \circ \rho_p^*]$$

for all $X \in \mathfrak{g}$ and all $Y \in \mathfrak{g}_p$, cf. (4.34), which follows directly from (3.5). \square

4.12 Invariant Metrics on Homogeneous Superspaces

In the standard theory a Riemannian metric on a G -homogeneous space M is called G -invariant if every $g \in G$ acts on M by isometries. The characterization of actions in the world of supermanifolds in terms of Harish-Chandra pairs immediately motivates the following definition:

A graded Riemannian metric on a G -homogeneous superspace M is *G-invariant* if every $g \in G_{\text{red}}$ acts on M by isometries and the image of the morphism $\mathfrak{g} \rightarrow T_M(M)^\circ$ lies in the subalgebra of graded Killing fields.

Theorem 4.23. *Let M be a G -homogeneous superspace, fix $p \in M_{\text{red}}$ and let $H = G_p$. Then there is a 1-1-correspondence between G -invariant graded Riemannian metrics on M and Ad_H -invariant scalar superproducts on the space $\mathfrak{g}/\mathfrak{h} \simeq T_H G/H \simeq T_p M$.*

Remark. By definition, Ad_H -invariance means $\text{Ad}_{H_{\text{red}}}$ - and $\text{ad}_{\mathfrak{h}}$ -invariance. If H is connected, $\text{ad}_{\mathfrak{h}}$ -invariance implies $\text{Ad}_{H_{\text{red}}}$ -invariance. Nevertheless, the converse is not true – $\text{Ad}_{H_{\text{red}}}$ -invariance only implies $\text{ad}_{\mathfrak{h}_0}$ -invariance.

Proof. First of all, take a G -invariant graded Riemannian metric $\langle \cdot, \cdot \rangle$ on M ; we have to show that the induced scalar superproduct on $\mathfrak{g}/\mathfrak{h}$ is $\text{Ad}_{H_{\text{red}}}$ - and $\text{ad}_{\mathfrak{h}}$ -invariant.

Since the action of any element $g \in G_{\text{red}}$ and thus in particular of any element $h \in H_{\text{red}}$ is isometric, Proposition 4.22 immediately shows that $\langle \cdot, \cdot \rangle_H$ is $\text{Ad}_{H_{\text{red}}}$ -invariant.

Let $X \in \mathfrak{h}$. If the action is denoted by ρ , then $\bar{X} := (X_e \otimes I) \circ \rho^*$ is a Killing field with value 0 at p . Pick $v, w \in T_p M$ and extend them to vector fields Y, Z on a neighbourhood of p . Then

$$\begin{aligned} 0 &= \bar{X} \langle Y, Z \rangle (p) = \langle [\bar{X}, Y], Z \rangle (p) + (-1)^{|\bar{X}||Y|} \langle Y, [\bar{X}, Z] \rangle (p) \\ &= \langle [\bar{X}, v], w \rangle_p + (-1)^{|\bar{X}||v|} \langle v, [\bar{X}, w] \rangle_p; \end{aligned}$$

note that the expressions in the second line are well-defined since \bar{X} vanishes at p . Translating the situation to $\mathfrak{g}/\mathfrak{h}$ with the help of Proposition 4.22 gives the $\text{ad}_{\mathfrak{h}}$ -invariance.

We now show the injectivity of the correspondence: Assume that there are G -invariant graded Riemannian metrics $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ that induce the same scalar superproduct on $T_p M$. The fact that every $g \in G_{\text{red}}$ acts on M by isometries with respect to both metrics immediately implies that they induce the same scalar superproduct on every tangent space. But that does not suffice to show that the metrics are equal. To achieve that, take two local vector fields Y, Z on M . We know that

$$\langle Y, Z \rangle_1 (q) = \langle Y, Z \rangle_2 (q)$$

for any q . Furthermore, for any Killing field X_1 ,

$$\begin{aligned} X_1 \langle Y, Z \rangle_1 (q) &= \langle [X_1, Y], Z \rangle_1 (q) + (-1)^{|X_1||Y|} \langle Y, [X_1, Z] \rangle_1 (q) \\ &= \langle [X_1, Y], Z \rangle_2 (q) + (-1)^{|X_1||Y|} \langle Y, [X_1, Z] \rangle_2 (q) = X_1 \langle Y, Z \rangle_2 (q). \end{aligned}$$

Applying further Killing fields X_2, \dots, X_n , we see analogously

$$X_n \dots X_1 \langle Y, Z \rangle_1 (q) = X_n \dots X_1 \langle Y, Z \rangle_2 (q),$$

since there only appear scalar products of some iterated Lie brackets, evaluated at q . The G -homogeneity of M now gives the existence of enough Killing fields to conclude $\langle Y, Z \rangle_1 = \langle Y, Z \rangle_2$.

It remains to show the surjectivity of the correspondence. Let $\langle \cdot, \cdot \rangle_H$ be an Ad_H -invariant scalar superproduct on $\mathfrak{g}/\mathfrak{h}$, which by Proposition 4.22 is the same as a scalar superproduct on $T_p M$, invariant under the isotropy representation of G_p , i.e. invariant under $(G_{\text{red}})_p$ and \mathfrak{g}_p , where the second condition is $\langle X \cdot v, w \rangle_p + (-1)^{|X||v|} \langle v, X \cdot w \rangle_p = 0$. The action of G_{red} now induces well-defined scalar superproducts $\langle \cdot, \cdot \rangle_q$ on $T_q M$ for all $q \in M_{\text{red}}$: for $g \in G_{\text{red}}$ with $gq = p$ we set $\langle v, w \rangle_q := \langle dg(v), dg(w) \rangle_p$. These are invariant under the action of the respective isotropy groups – on the level of Lie groups this follows directly from $(G_{\text{red}})_q = g^{-1} \cdot (G_{\text{red}})_p \cdot g$; on the level of Lie superalgebras we calculate for $X \in \mathfrak{g}_q$ as follows:

$$\begin{aligned} \langle X \cdot v, w \rangle_q &= -\langle [\bar{X}, v], w \rangle_q = -\langle dg[\bar{X}, v], dg(w) \rangle_p \\ &= -\langle [dg(\bar{X}), dg(v)], dg(w) \rangle_p = (-1)^{|X||v|} \langle dg(v), [dg(\bar{X}), dg(w)] \rangle_p \\ &= -(-1)^{|X||v|} \langle v, X \cdot w \rangle_q. \end{aligned}$$

Let X_1, \dots, X_n be a basis of the Lie algebra \mathfrak{g} and consider the associated vector fields $\bar{X}_i = ((X_i)_e \otimes I) \circ \rho^*$. If (η_i) are coordinates on some open set U , we can express ∂_j as a linear combination of the \bar{X}_i , since M is G -homogeneous:

$$\partial_j = \sum_i f_i^j \bar{X}_i. \quad (4.36)$$

We may assume that

$$|\partial_j| = |f_i^j| + |\bar{X}_i| \quad (4.37)$$

for all i and j . Of course, there is no unique way of doing so, but if

$$\sum_i f_i \bar{X}_i = \sum_i g_i \bar{X}_i \quad (4.38)$$

are two representations of the same vector field, we have

$$\sum_i (f_i(q) - g_i(q)) X_i \in \mathfrak{g}_q$$

for all $q \in U$ by definition of \mathfrak{g}_q (4.33).

Distinguish now between even and odd coordinates x_i and ξ_α and take two vector fields Y and Z on U . Our strategy is as follows: Under the assumption of its existence we compute $\langle Y, Z \rangle$ in terms of the scalar superproducts at all points $q \in U$; then we show that the resulting expressions provide us with a well-defined G -invariant metric.

$$\langle Y, Z \rangle = \langle Y, Z \rangle(q) + \sum_\alpha (\partial_\alpha \langle Y, Z \rangle)(q) \xi_\alpha + \sum_{\alpha < \beta} (\partial_\alpha \partial_\beta \langle Y, Z \rangle)(q) \xi_\beta \xi_\alpha + \dots \quad (4.39)$$

Choosing some representation of ∂_α in terms of the \bar{X}_i , (4.36), we calculate

$$(\partial_\alpha \langle Y, Z \rangle)(q) = \sum_i f_i^\alpha(q) \left(\langle [\bar{X}_i, Y], Z \rangle(q) + (-1)^{|\bar{X}_i||Y|} \langle Y, [\bar{X}_i, Z] \rangle(q) \right).$$

The right-hand side now defines a \mathcal{C}^∞ -function $G_\alpha^{Y,Z}(q)$; it is independent of the representation of ∂_α in the \bar{X}_i since the scalar superproduct at q is G_q -invariant.

Again assuming the existence of $\langle Y, Z \rangle$, we compute the part of degree two:

$$\begin{aligned} & (\partial_\alpha \partial_\beta \langle Y, Z \rangle)(q) \\ &= \partial_\alpha \left(\sum_i f_i^\beta \left(\langle [\bar{X}_i, Y], Z \rangle + (-1)^{|\bar{X}_i||Y|} \langle Y, [\bar{X}_i, Z] \rangle \right) \right)(q) \\ &= \sum_i (\partial_\alpha f_i^\beta)(q) \left(\langle [\bar{X}_i, Y], Z \rangle(q) + (-1)^{|\bar{X}_i||Y|} \langle Y, [\bar{X}_i, Z] \rangle(q) \right) \\ &\quad + \sum_{i,j} (-1)^{|f_i^\beta|} f_i^\beta(q) f_j^\alpha(q) \left(\langle [\bar{X}_j, [\bar{X}_i, Y]], Z \rangle(q) \right. \\ &\quad + (-1)^{|\bar{X}_j|(|\bar{X}_i|+|Y|)} \langle [\bar{X}_i, Y], [\bar{X}_j, Z] \rangle(q) + (-1)^{|\bar{X}_i||Y|} \langle [\bar{X}_j, Y], [\bar{X}_i, Z] \rangle(q) \\ &\quad \left. + (-1)^{|Y|(|\bar{X}_i|+|\bar{X}_j|)} \langle Y, [\bar{X}_j, [\bar{X}_i, Z]] \rangle(q) \right) \end{aligned}$$

We define a \mathcal{C}^∞ -function $G_{\alpha\beta}^{Y,Z}(q)$ by this expression and have to show that it is independent of the choice of representation (4.36). Taking a second representation as in (4.38) we have

$$\begin{aligned} 0 &= [\partial_\beta, \partial_\alpha] = \partial_\beta \circ \partial_\alpha + \partial_\alpha \circ \partial_\beta \\ &= \left(\sum_i f_i^\beta \bar{X}_i \right) \left(\sum_j g_j^\alpha \bar{X}_j \right) + \left(\sum_j g_j^\alpha \bar{X}_j \right) \left(\sum_i f_i^\beta \bar{X}_i \right) \\ &= \sum_j (\partial_\beta g_j^\alpha) \bar{X}_j + \sum_i (\partial_\alpha f_i^\beta) \bar{X}_i + \sum_{i,j} (-1)^{|\bar{X}_i| |g_j^\alpha|} f_i^\beta g_j^\alpha [\bar{X}_i, \bar{X}_j], \end{aligned}$$

where we used the parity convention (4.37). In other words,

$$\sum_i \left((\partial_\beta g_j^\alpha)(q) + (\partial_\alpha f_i^\beta)(q) \right) X_i + \sum_{i,j} (-1)^{|X_i|} f_i^\beta(q) g_j^\alpha(q) [X_i, X_j] \in \mathfrak{g}_q \quad (4.40)$$

for all q . Using the G_q -invariance of the scalar superproduct at q , a short calculation shows that $G_{\alpha\beta}^{Y,Z}$ is independent of the choice of representation.

Similar but longer calculations give well-defined smooth functions $G_{\alpha_1 \dots \alpha_k}^{Y,Z}$ for any k -tuple $\alpha_1 < \dots < \alpha_k$. Then we can define a G -invariant metric by (4.39). \square

4.13 Riemannian Symmetric Superspaces

A conventional Riemannian manifold M is called a Riemannian symmetric space if for every point $p \in M$ there exists an isometry s_p of M with $s_p(p) = p$ and $d_p s_p = -\text{id}_{T_p M}$. Translating this into the world of supermanifolds we arrive at the following definition deviating from the usual one by the additional infinitesimal odd part: A Riemannian supermanifold M is called *symmetric* or a *(Riemannian) symmetric superspace* if for any point p there exists an isometry s_p of M with $s_p(p) = p$ and $d_p s_p = -\text{id}_{T_p M}$ and if for any odd tangent vector $\tau \in (T_p M)_1$ there exists a Killing vector field S_τ on M with $(S_\tau)_p = \tau$ and $(\nabla S_\tau)(p) = 0$. In the standard theory, the Killing vector fields X with $(\nabla X)(p) = 0$ are exactly those defined by transvections (which in turn are constructed via the geodesic symmetries) so the existence of Killing fields of this type is the correct infinitesimal counterpart of the existence of the geodesic symmetries.

Remark. Infinitesimal versions of this definition already exist in the mathematical literature, see e.g. [Cor 2003] or [Ser 1983].

Just like in the standard theory, a symmetric superspace is homogeneous (the surjectivity of the mappings $\{\text{odd Killing fields}\} \rightarrow (T_p M)_1$ is trivially fulfilled).

Let M be a symmetric superspace and set $G = I_0(M)$, the identity component of the isometry group of M . Let $K = G_p$, the isotropy group of some point p . Conjugation with s_p induces a morphism $\sigma : G \rightarrow G$ with $d\sigma = \text{Ad}_{s_p} : \mathfrak{g} \rightarrow \mathfrak{g}$. Clearly, σ is involutive.

We define the fixed point group G^σ of σ to be the Lie subsupergroup of G given by the Harish-Chandra pair $(G_{\text{red}}^\sigma, \mathfrak{g}^\sigma)$ with $G_{\text{red}}^\sigma = \{x \in G_{\text{red}} \mid \sigma(x) = x\}$ and $\mathfrak{g}^\sigma = \{X \in \mathfrak{g} \mid d\sigma(X) = X\}$.

Lemma 4.24. *Under these conditions, we have $G_0^\sigma \subset K \subset G^\sigma$.*

Proof. The first inclusion can be verified on the level of Lie algebras: If X is a Killing field on M such that $d\sigma(X) = X$, i.e. $ds_p(X) = X$, the value of X at p clearly has to vanish since $d_p s_p = -\text{id}$.

Let $\phi \in K_{\text{red}}$. Then ϕ and $\sigma(\phi) = s_p \circ \phi \circ s_p$ are isometries of M sending p to itself and having the same differential at p ; because of Proposition 4.11, they are equal, i.e. $\phi \in G_{\text{red}}^\sigma$.

We have to show that \mathfrak{k} , the Lie algebra of K , is contained in \mathfrak{g}^σ , so let $X \in \mathfrak{g}$ be a Killing field that vanishes at p . Then $ds_p(X)$ is a Killing field also vanishing at p and satisfying

$$(\nabla_Y X)(p) = -d_p s_p((\nabla_Y X)(p)) = -(\nabla_{ds_p(Y)} ds_p(X))(p) = (\nabla_Y ds_p(X))(p) \quad (4.41)$$

for all Y ; Proposition 4.14 thus yields $X = ds_p(X)$. \square

Since $d\sigma$ is an involutive automorphism, \mathfrak{g} splits as the sum of its $(+1)$ - and (-1) -eigenspace – we can write any $X \in \mathfrak{g}$ as $X = \frac{1}{2}(X + d\sigma(X)) + \frac{1}{2}(X - d\sigma(X))$. We showed that the $+1$ -eigenspace coincides with the Lie algebra of K , so

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with $\mathfrak{p} = \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}$. The usual relations $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$, $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ hold.

Lemma 4.25. *The space \mathfrak{p} is the space of all Killing vector fields X on M such that $(\nabla X)(p) = 0$.*

Proof. Let X be a Killing vector field with $(\nabla X)(p) = 0$. Then $d\sigma(X)$ and $-X$ are Killing fields having the same value and by (4.41) the same derivative at p ; Proposition 4.14 thus yields $X \in \mathfrak{p}$.

If conversely $X \in \mathfrak{p}$ is given, (4.41) immediately shows $(\nabla_Y X)(p) = 0$ for all Y . \square

Corollary 4.26. *A Riemannian supermanifold M is symmetric if and only if it is homogeneous and there exists a point $p \in M$ with an isometry $s_p : M \rightarrow M$ leaving p fixed and satisfying $d_p s_p = -\text{id}_{T_p M}$.*

Let G be a connected Lie supergroup and K a closed Lie subsupergroup. Then the pair (G, K) is a *symmetric pair* if there exists an involutive automorphism σ of G such that $G_0^\sigma \subset K \subset G^\sigma$, where G^σ is the group of fixed points of σ .

Proposition 4.27. *Let (G, K) be a symmetric pair and $\langle \cdot, \cdot \rangle$ a G -invariant graded Riemannian metric on G/K . Then G/K is a Riemannian symmetric superspace.*

Remark. We do not state that such an invariant metric always exists!

Proof. Let $s_K : G/K \rightarrow G/K$ be the morphism induced by σ . More precisely, for $f \in \mathcal{O}_{G/K}(U)$ we set $s_K^*(f) := \sigma^*(f)$; this is well-defined because of $\sigma \circ \Phi = \sigma \circ \text{pr}_1$, where the morphisms Φ and pr_1 are the same mappings as in (4.32). Then

$$s_K \circ \pi = \pi \circ \sigma,$$

where $\pi : G \rightarrow G/K$ is the canonical projection. Differentiating this equation at $e \in G$, we get $d_K s_K = -\text{id}_{T_K G/K}$. In view of Corollary 4.26, it remains to show that s_K is an isometry.

For any g we will write $g : G/K \rightarrow G/K$ for left translation with g . Let $p = gK_{\text{red}} \in G_{\text{red}}/K_{\text{red}}$, pick vector fields X and Y around p and define $X_0 := dg^{-1}(X)$ and $Y_0 := dg^{-1}(Y)$. If $\rho : G \times G/K \rightarrow G/K$ is the standard action, we have

$$s_K \circ \rho = \rho \circ (\sigma \times s_K) \quad (4.42)$$

and thus in particular $s_K \circ g = \sigma(g) \circ s_K$. This yields

$$\begin{aligned} \langle ds_K(X_p), ds_K(Y_p) \rangle_{s_K(p)} &= \langle ds_K(dg(X_{0,K})), ds_K(dg(Y_{0,K})) \rangle_{s_K(p)} \\ &= \langle d(\sigma(g))(ds_K(X_{0,K})), d(\sigma(g))(ds_K(Y_{0,K})) \rangle_{s_K(p)} \\ &= \langle d(\sigma(g))(X_{0,K}), d(\sigma(g))(Y_{0,K}) \rangle_{s_K(p)} \\ &= \langle X_{0,K}, Y_{0,K} \rangle_K = \langle X_p, Y_p \rangle_p, \end{aligned} \quad (4.43)$$

where we used the G -invariance of the metric for the last two equalities.

For any $Z \in \mathfrak{g}$, the vector field $\bar{Z} = (Z_e \otimes I) \circ \rho^*$ is a Killing field because of the G -invariance of the metric. Then

$$\begin{aligned} ds_K(\bar{Z}) &= s_K^* \circ (Z_e \otimes I) \circ (s_K \circ \rho)^* = s_K^* \circ (X_e \otimes I) \circ (\sigma \times s_K)^* \circ \rho^* \\ &= (d\sigma(Z)_e \otimes I) \circ \rho^* = \overline{d\sigma(Z)}, \end{aligned}$$

so in particular $ds_K(\bar{Z})$ is a Killing field. Thus we may calculate

$$\begin{aligned} ds_K(\bar{Z}) \langle ds_K(X), ds_K(Y) \rangle (s_K(p)) &= \langle [ds_K(\bar{Z}), ds_K(X)]_{s_K(p)}, ds_K(Y_p) \rangle_{s_K(p)} \\ &\quad + (-1)^{|\bar{Z}||X|} \langle ds_K(X_p), [ds_K(\bar{Z}), ds_K(Y)]_{s_K(p)} \rangle_{s_K(p)} \\ &= \langle [\bar{Z}, X]_p, Y_p \rangle_p + (-1)^{|\bar{Z}||X|} \langle X_p, [\bar{Z}, Y]_p \rangle_p = \bar{Z} \langle X, Y \rangle (p), \end{aligned}$$

where we used (4.43). The same calculation shows

$$ds_K(\bar{Z}_1) \circ \dots \circ ds_K(\bar{Z}_n) \langle ds_K(X), ds_K(Y) \rangle (s_K(p)) = \bar{Z}_1 \circ \dots \circ \bar{Z}_n \langle X, Y \rangle (p)$$

for all $Z_1, \dots, Z_n \in \mathfrak{g}$. The G -invariance of the metric, together with the G -homogeneity of G/K shows that there are enough Killing vector fields induced by the action to conclude $s_K^* \langle ds_K(X), ds_K(Y) \rangle = \langle X, Y \rangle$. \square

4.14 Examples

The trivial examples are the following: Clearly, any Riemannian symmetric space is a Riemannian symmetric superspace. Furthermore, $\mathbf{R}^{p|2q}$ with the standard metric is a Riemannian symmetric superspace. Thus, the exterior bundle of the trivial bundle $M \times \mathbf{R}^{2q} \rightarrow M$ over a Riemannian symmetric space M gives rise to a Riemannian symmetric superspace since in our language it is merely $M \times \mathbf{R}^{0|2q}$, the product of two such spaces.

Just like in the standard theory, groups that admit bi-invariant metrics give a class of examples.

Proposition 4.28. *A Lie supergroup with a bi-invariant graded Riemannian metric is a Riemannian symmetric superspace.*

Proof. Let G be a Lie supergroup with a bi-invariant graded Riemannian metric. Since the metric is left-invariant, G is homogeneous. The bi-invariance shows that the inverse map i is an isometry (it interchanges left-invariant vector fields with right-invariant vector fields and its differential $di : T_e G \rightarrow T_e G$ is minus the identity) and thus gives the symmetry at e . \square

Inspired by the Zirnbauer list [Zir 1996], we now present examples of series of Riemannian symmetric superspaces. Note that his notion of Riemannian symmetric superspace is in so far different from ours since he defines them as quotients of *complex* Lie supergroups with a certain additional condition. We emphasize that the following list is no attempt of a complete classification – more examples can for example be given by duality. The classification problem is related to the problem of classifying pseudo-Riemannian symmetric spaces [CahPar 1980], which is solved in the semi-simple case [Ber 1957] but e.g. not in the solvable case, see [KatOlb 2004].

4.14.1 The RSSS $\mathrm{SL}(n|2m)/\mathrm{SOSp}(n|2m)$ ($n \neq 2m$)

Consider on the Lie superalgebra $\mathfrak{sl}(n|2m)$ the involution σ , given by

$$\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_1 & D_2 \\ C_2 & D_3 & D_4 \end{array} \right) \mapsto \left(\begin{array}{c|cc} -A^t & C_2^t & -C_1^t \\ \hline -B_2^t & -D_4^t & D_2^t \\ B_1^t & D_3^t & -D_1^t \end{array} \right),$$

where A, B_i, C_i and D_i are $n \times n$ -, $n \times m$ -, $m \times n$ - and $m \times m$ -matrices, respectively. An easy calculation shows that σ is an automorphism. The decomposition into the eigenspaces of σ is

$$\mathfrak{sl}(n|2m) = \mathfrak{osp}(n|2m) \oplus \mathfrak{p}$$

$$\text{with } \mathfrak{p} = \left\{ \left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline B_2^t & D_1 & D_2 \\ -B_1^t & D_3 & D_1^t \end{array} \right) \in \mathfrak{sl}(n|2m) \mid A^t = A, D_2^t = -D_2, D_3^t = -D_3 \right\}.$$

The involution σ is induced by an isometry of $\mathrm{SL}(n|2m)$: On the level of the underlying Lie groups, the involutive automorphism of $\mathrm{GL}(n) \times \mathrm{GL}(2m)$ given by

$$(X, Y) \mapsto ((X^{-1})^t, -J_m(Y^{-1})^t J_m),$$

where $J_m = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}$, restricts to an involutive automorphism of the reduced group $\mathrm{SL}(n|2m)_{\mathrm{red}}$ with σ_0 as differential and $\mathrm{SO}(n) \times \mathrm{Sp}(m, \mathbf{R})$ as (connected) fixed point group. Since the question of extending a morphism of Lie superalgebras to a morphism of Lie supergroups concerns only the underlying Lie group ([DelMor 1999], p. 69), σ is induced by an involutive automorphism of $\mathrm{SL}(n|2m)$ (which is also denoted by σ). Thus, $(\mathrm{SL}(n|2m), \mathrm{SOSp}(n|2m))$ is a symmetric pair.

In view of Proposition 4.27 and Theorem 4.23, we have to find an $\mathrm{Ad}_{\mathrm{SL}(n|2m)}$ -invariant scalar superproduct on \mathfrak{p} . Clearly, the supertrace induces an invariant supersymmetric bilinear form $(X, Y) \mapsto \mathrm{str}(XY)$; we have to show its non-degeneracy. To show the non-degeneracy of its even part, note that the even part of $\mathfrak{sl}(n|2m)$ splits as $\mathfrak{sl}(n) \oplus \mathfrak{sl}(2m) \oplus \mathfrak{u}(1)$. Then

$$\mathfrak{p}_0 = \mathfrak{p}_0 \cap \mathfrak{sl}(n) \oplus \mathfrak{p}_0 \cap \mathfrak{sl}(2m) \oplus \mathfrak{u}(1)$$

is an orthogonal decomposition with respect to the supertrace. On the first two summands, the supertrace clearly is non-degenerate; on the third, it is so because $n \neq 2m$. To prove the non-degeneracy of the odd part, we define for

$$0 \neq X = \left(\begin{array}{c|cc} 0 & B_1 & B_2 \\ \hline B_2^t & 0 & 0 \\ -B_1^t & 0 & 0 \end{array} \right) \in \mathfrak{p}_1$$

an element

$$Y = \left(\begin{array}{c|cc} 0 & -B_2 & B_1 \\ \hline B_1^t & 0 & 0 \\ B_2^t & 0 & 0 \end{array} \right) \in \mathfrak{p}_1$$

to get $\mathrm{str}(XY) = 2(\mathrm{tr}(B_1 B_1^t) + \mathrm{tr}(B_2 B_2^t)) > 0$. We have thus shown that $\mathrm{SL}(n|2m)/\mathrm{SOSp}(n|2m)$, with the metric induced by the supertrace, is a Riemannian symmetric superspace for $n \neq 2m$. Note that the Killing form is a non-zero multiple of the supertrace since $n \neq 2m$ (see 4.10) so in this example the Killing form also gives an invariant metric.

4.14.2 The RSSS $\mathrm{PSL}(2m|2m)/\mathrm{SOSp}(2m|2m)$

The automorphism σ , defined as in the previous example, induces an automorphism of $\mathfrak{psl}(2m|2m)$. Then the same argumentation as above introduces the structure of a Riemannian symmetric superspace on $\mathrm{PSL}(2m|2m)/\mathrm{SOSp}(2m|2m)$; by passing to the quotient $\mathrm{PSL}(2m|2m)$ of $\mathrm{SL}(2m|2m)$ we achieve that the supertrace is non-degenerate. Note that here we do not have a metric induced by the Killing form since the Killing form of $\mathfrak{sl}(2m|2m)$ vanishes, cf. 4.10.

4.14.3 The RSSS $\mathrm{SL}(n_1 + n_2|m_1 + m_2)/\mathrm{S}(\mathrm{GL}(n_1|m_1) \times \mathrm{GL}(n_2|m_2))$

The involution σ on the Lie superalgebra $\mathfrak{sl}(n_1 + n_2|m_1 + m_2)$ given by

$$\left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_3 & A_4 & B_3 & B_4 \\ \hline C_1 & C_2 & D_1 & D_2 \\ C_3 & C_4 & D_3 & D_4 \end{array} \right) \mapsto \left(\begin{array}{cc|cc} A_1 & -A_2 & B_1 & -B_2 \\ -A_3 & A_4 & -B_3 & B_4 \\ \hline C_1 & -C_2 & D_1 & -D_2 \\ -C_3 & C_4 & -D_3 & D_4 \end{array} \right)$$

is an automorphism. The corresponding decomposition of $\mathfrak{sl}(n_1 + n_2|m_1 + m_2)$ is

$$\mathfrak{sl}(n_1 + n_2|m_1 + m_2) = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\begin{aligned} \mathfrak{k} &= \mathfrak{s}(\mathfrak{gl}(n_1|m_1) \times \mathfrak{gl}(n_2|m_2)) \\ &= \{(X, Y) \in \mathfrak{gl}(n_1|m_1) \times \mathfrak{gl}(n_2|m_2) \mid \text{str}(X) + \text{str}(Y) = 0\} \end{aligned}$$

is embedded into $\mathfrak{sl}(n_1 + n_2|m_1 + m_2)$ via

$$\left(\left(\frac{A}{C} \middle| \frac{B}{D} \right), \left(\frac{A'}{C'} \middle| \frac{B'}{D'} \right) \right) \mapsto \left(\frac{A \quad 0}{0 \quad A'} \middle| \frac{B \quad 0}{D \quad 0} \right)$$

and the space \mathfrak{p} is given by

$$\mathfrak{p} = \left\{ \left(\frac{0 \quad A_1}{A_2 \quad 0} \middle| \frac{0 \quad B_1}{B_2 \quad 0} \right) \in \mathfrak{sl}(n_1 + n_2|m_1 + m_2) \right\}.$$

On the level of Lie groups,

$$(X, Y) \mapsto (I_{n_1, n_2} X I_{n_1, n_2}, I_{m_1, m_2} Y I_{m_1, m_2}),$$

where $I_{n, m} = \begin{pmatrix} -I_n & 0 \\ 0 & I_m \end{pmatrix}$, is an involution of $\text{GL}(n_1 + n_2) \times \text{GL}(m_1 + m_2)$ which restricts to an involution of $\text{SL}(n_1 + n_2|m_1 + m_2)_{\text{red}}$ with fixed point group $\text{S}(\text{GL}(n_1) \times \text{GL}(n_2) \times \text{GL}(m_1) \times \text{GL}(m_2))$ and differential σ_0 . Thus, σ is induced by an involution of $\text{SL}(n_1 + n_2|m_1 + m_2)$. Defining the Lie supergroup $\text{S}(\text{GL}(n_1|m_1) \times \text{GL}(n_2|m_2))$ to be given by the Harish-Chandra pair

$$(\text{S}(\text{GL}(n_1) \times \text{GL}(m_1) \times \text{GL}(n_2) \times \text{GL}(m_2)), \mathfrak{s}(\mathfrak{gl}(n_1|m_1) \times \mathfrak{gl}(n_2|m_2))),$$

we thus see that $(\text{SL}(n_1 + n_2|m_1 + m_2), \text{S}(\text{GL}(n_1|m_1) \times \text{GL}(n_2|m_2)))$ is a symmetric pair.

The supertrace $(X, Y) \mapsto \text{str}(XY)$ is an invariant scalar superproduct on \mathfrak{p} ; the induced invariant metric gives the structure of a Riemannian symmetric superspace.

Note that for $n_1 + n_2 = m_1 + m_2 =: n$, the Killing form vanishes and thus would not be appropriate. In this case, note also that we may write this Riemannian symmetric superspace as

$$\text{PSL}(n|n)/\text{PS}(\text{GL}(n_1|m_1) \times \text{GL}(n_2|m_2)),$$

where the P in $\text{PS}(\text{GL}(n_1|m_1) \times \text{GL}(n_2|m_2))$ means passing to the Lie supergroup obtained by factoring out the one-dimensional center generated by the identity matrix.

4.14.4 The RSSS $\text{SOSp}(2n|2m)/\text{U}(n|m)$

Write the elements of $\mathfrak{osp}(2n|2m)$ in the form

$$\left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ -A_2^t & A_3 & B_3 & B_4 \\ \hline -B_2^t & -B_4^t & C_1 & C_2 \\ B_1^t & B_3^t & C_3 & -C_1^t \end{array} \right),$$

where the A_i, B_i and C_i are $n \times n$ -, $n \times m$ - and $m \times m$ -matrices, respectively, satisfying the relations $A_1^t = -A_1, A_3^t = -A_3, C_2^t = C_2$ and $C_3^t = C_3$.

The involutive automorphism σ of $\mathfrak{osp}(2n|2m)$, given by

$$\left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ -A_2^t & A_3 & B_3 & B_4 \\ \hline -B_2^t & -B_4^t & C_1 & C_2 \\ B_1^t & B_3^t & C_3 & -C_1^t \end{array} \right) \mapsto \left(\begin{array}{cc|cc} A_3 & A_2^t & B_4 & -B_3 \\ -A_2 & A_1 & -B_2 & B_1 \\ \hline B_3^t & -B_1^t & -C_1^t & -C_3 \\ B_4^t & -B_2^t & -C_2 & C_1 \end{array} \right),$$

yields the decomposition

$$\mathfrak{osp}(2n|2m) = \mathfrak{k} \oplus \mathfrak{p},$$

where

$$\mathfrak{k} = \left\{ \left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ -A_2 & A_1 & -B_2 & B_1 \\ \hline -B_2^t & -B_1^t & C_1 & C_2 \\ B_1^t & -B_2^t & -C_2 & C_1 \end{array} \right) \mid A_1^t = -A_1, A_2^t = A_2, C_1^t = -C_1, C_2^t = C_2 \right\}$$

and

$$\mathfrak{p} = \left\{ \left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ A_2 & -A_1 & B_2 & -B_1 \\ \hline -B_2^t & B_1^t & C_1 & C_2 \\ B_1^t & B_2^t & C_2 & -C_1 \end{array} \right) \mid A_1^t = -A_1, A_2^t = -A_2, C_1^t = C_1, C_2^t = C_2 \right\}$$

Recall that the unitary superalgebra $\mathfrak{u}(n|m)$ is defined as

$$\mathfrak{u}(n|m) = \left\{ \left(\begin{array}{c|c} A & B \\ \hline -iB^* & C \end{array} \right) \mid A, B, C \text{ complex}, A^* = -A, C^* = -C \right\}.$$

We can identify it with \mathfrak{k} via

$$\left(\begin{array}{c|c} A_1 + iA_2 & B_1 + iB_2 \\ \hline -B_2^t - iB_1^t & C_1 + iC_2 \end{array} \right) \mapsto \left(\begin{array}{cc|cc} A_1 & A_2 & B_1 & B_2 \\ -A_2 & A_1 & -B_2 & B_1 \\ \hline -B_2^t & -B_1^t & C_1 & C_2 \\ B_1^t & -B_2^t & -C_2 & C_1 \end{array} \right),$$

where the A_i, B_i and C_i are real matrices.

On the level of Lie groups,

$$(X, Y) \mapsto (-J_n X J_n, -J_m Y J_m)$$

is an automorphism of $\mathrm{SO}(2n) \times \mathrm{Sp}(m; \mathbf{R})$ with differential σ_0 and fixed point group isomorphic to $\mathrm{U}(n) \times \mathrm{U}(m)$. Thus, there is an automorphism of the Lie supergroup $\mathrm{SOSp}(2n|2m)$ turning $(\mathrm{SOSp}(2n|2m), \mathrm{U}(n|m))$ into a symmetric pair.

This example somehow plays an extraordinary role since on \mathfrak{p}_0 , the supertrace is negative definite:

$$\mathrm{str} \left(\begin{array}{cc|cc} A_1 & A_2 & 0 & 0 \\ A_2 & -A_1 & 0 & 0 \\ \hline 0 & 0 & C_1 & C_2 \\ 0 & 0 & C_2 & -C_1 \end{array} \right)^2 = 2 \mathrm{tr}(A_1^2 + A_2^2) - 2 \mathrm{tr}(C_1^2 + C_2^2).$$

It is non-degenerate on the odd part since for any non-vanishing

$$X = \left(\begin{array}{cc|cc} 0 & 0 & B_1 & B_2 \\ 0 & 0 & B_2 & -B_1 \\ \hline -B_2^t & B_1^t & 0 & 0 \\ B_1^t & B_2^t & 0 & 0 \end{array} \right) \in \mathfrak{p}_1$$

we define

$$Y = \left(\begin{array}{cc|cc} 0 & 0 & B_2 & -B_1 \\ 0 & 0 & -B_1 & -B_2 \\ \hline B_1^t & B_2^t & 0 & 0 \\ B_2^t & -B_1^t & 0 & 0 \end{array} \right) \in \mathfrak{p}_1$$

to get

$$\mathrm{str}(XY) = 4 \mathrm{tr}(B_1 B_1^t + B_2 B_2^t) > 0.$$

Thus, equipped with the metric induced by the negative of the supertrace, $\mathrm{SOSp}(2n|2m)/\mathrm{U}(n|m)$ becomes a Riemannian symmetric superspace such that the reduced manifold is a Riemannian symmetric space (the product of one of compact type and one of non-compact type, $\mathrm{SO}(2n)/\mathrm{U}(n) \times \mathrm{Sp}(m; \mathbf{R})/\mathrm{U}(m)$).

4.14.5 The RSSS $\mathrm{SOSp}(n_1+n_2|2m_1+2m_2)/\mathrm{S}(\mathrm{OSp}(n_1|2m_1) \times \mathrm{OSp}(n_2|2m_2))$

Writing the elements of $\mathfrak{osp}(n_1 + n_2|2m_1 + 2m_2)$ in the form

$$\left(\begin{array}{cc|cccc} A_{11} & A_{12} & B_{11} & B_{12} & B_{13} & B_{14} \\ -A_{12}^t & A_{22} & B_{21} & B_{22} & B_{23} & B_{24} \\ \hline -B_{13}^t & -B_{23}^t & C_{11} & C_{12} & C_{13} & C_{14} \\ -B_{14}^t & -B_{24}^t & C_{21} & C_{22} & C_{14}^t & C_{24} \\ B_{11}^t & B_{21}^t & C_{31} & C_{32} & -C_{11}^t & -C_{21}^t \\ B_{12}^t & B_{22}^t & C_{32}^t & C_{42} & -C_{12}^t & -C_{22}^t \\ \underbrace{\hspace{1.5cm}}_{n_1} & \underbrace{\hspace{1.5cm}}_{n_2} & \underbrace{\hspace{1.5cm}}_{m_1} & \underbrace{\hspace{1.5cm}}_{m_2} & \underbrace{\hspace{1.5cm}}_{m_3} & \underbrace{\hspace{1.5cm}}_{m_4} \end{array} \right)$$

with $A_{11} = -A_{11}^t$, $A_{22} = -A_{22}^t$, $C_{13} = C_{13}^t$, $C_{31} = C_{31}^t$, $C_{24} = C_{24}^t$ and $C_{42} = C_{42}^t$, we define an involutive automorphism σ of $\mathfrak{osp}(n_1 + n_2|2m_1 + 2m_2)$

as follows:

$$\begin{pmatrix} A_{11} & A_{12} & B_{11} & B_{12} & B_{13} & B_{14} \\ -A_{12}^t & A_{22} & B_{21} & B_{22} & B_{23} & B_{24} \\ -B_{13}^t & -B_{23}^t & C_{11} & C_{12} & C_{13} & C_{14} \\ -B_{14}^t & -B_{24}^t & C_{21} & C_{22} & C_{14}^t & C_{24} \\ B_{11}^t & B_{21}^t & C_{31} & C_{32} & -C_{11}^t & -C_{21}^t \\ B_{12}^t & B_{22}^t & C_{32}^t & C_{42} & -C_{12}^t & -C_{22}^t \end{pmatrix} \mapsto \begin{pmatrix} A_{11} & -A_{12} & B_{11} & -B_{12} & B_{13} & -B_{14} \\ A_{12}^t & A_{22} & -B_{21} & B_{22} & -B_{23} & B_{24} \\ -B_{13}^t & B_{23}^t & C_{11} & -C_{12} & C_{13} & -C_{14} \\ B_{14}^t & -B_{24}^t & -C_{21} & C_{22} & -C_{14}^t & C_{24} \\ B_{11}^t & -B_{21}^t & C_{31} & -C_{32} & -C_{11}^t & C_{21}^t \\ -B_{12}^t & B_{22}^t & -C_{32}^t & C_{42} & C_{12}^t & -C_{22}^t \end{pmatrix}.$$

The induced decomposition is

$$\mathfrak{osp}(n_1 + n_2 | 2m_1 + 2m_2) = (\mathfrak{osp}(n_1 | 2m_1) \oplus \mathfrak{osp}(n_2 | 2m_2)) \oplus \mathfrak{p},$$

where $\mathfrak{osp}(n_1 | 2m_1) \oplus \mathfrak{osp}(n_2 | 2m_2)$ is embedded into $\mathfrak{osp}(n_1 + n_2 | 2m_1 + 2m_2)$ via

$$\left(\left(\begin{array}{c|cc} A & B_1 & B_2 \\ \hline -B_2^t & C_1 & C_2 \\ B_1^t & C_3 & -C_1^t \end{array} \right), \left(\begin{array}{c|cc} A' & B'_1 & B'_2 \\ \hline -B_2'^t & C'_1 & C'_2 \\ B_1'^t & C'_3 & -C_1'^t \end{array} \right) \right) \mapsto \begin{pmatrix} A & 0 & B_1 & 0 & B_2 & 0 \\ 0 & A' & 0 & B'_1 & 0 & B'_2 \\ -B_2'^t & 0 & C_1 & 0 & C_2 & 0 \\ 0 & -B_2'^t & 0 & C'_1 & 0 & C'_2 \\ B_1^t & 0 & C_3 & 0 & -C_1^t & 0 \\ 0 & B_1'^t & 0 & C'_3 & 0 & -C_1'^t \end{pmatrix}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & A_{12} & 0 & B_{12} & 0 & B_{14} \\ -A_{12}^t & 0 & B_{21} & 0 & B_{23} & 0 \\ 0 & -B_{23}^t & 0 & C_{12} & 0 & C_{14} \\ -B_{14}^t & 0 & C_{21} & 0 & C_{14}^t & 0 \\ 0 & B_{21}^t & 0 & C_{32} & 0 & -C_{21}^t \\ B_{12}^t & 0 & C_{32}^t & 0 & -C_{12}^t & 0 \end{pmatrix} \in \mathfrak{osp}(n_1 + n_2 | 2m_1 + 2m_2) \right\}.$$

On the level of Lie groups,

$$(X, Y) \mapsto (I_{n_1, n_2} X I_{n_1, n_2}, L_{m_1, m_2} Y L_{m_1, m_2}),$$

where $L_{n, m} = \begin{pmatrix} -I_n & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & I_m \end{pmatrix}$, is an automorphism of the reduced

Lie group $\mathrm{SO}(n_1 + n_2) \times \mathrm{Sp}(n; \mathbf{R})$ with fixed point group

$$\mathrm{S}(\mathrm{O}(n_1) \times \mathrm{O}(n_2)) \times \mathrm{Sp}(m_1; \mathbf{R}) \times \mathrm{Sp}(m_2; \mathbf{R})$$

and differential σ_0 . Via the corresponding automorphism of the Lie supergroup $\text{SOSp}(n_1+n_2|2m_1+2m_2)$, the pair $(\text{SOSp}(n_1+n_2|2m_1+2m_2), \text{S}(\text{OSp}(n_1|2m_1) \times \text{OSp}(n_2|2m_2)))$ becomes a symmetric pair – here, $\text{S}(\text{OSp}(n_1|2m_1) \times \text{OSp}(n_2|2m_2))$ is the connected component of $\text{OSp}(n_1|2m_1) \times \text{OSp}(n_2|2m_2)$. Again, the supertrace induces an invariant metric on the corresponding homogeneous superspace $\text{SOSp}(n_1+n_2|2m_1+2m_2)/\text{S}(\text{OSp}(n_1|2m_1) \times \text{OSp}(n_2|2m_2))$ and thus turns it into a Riemannian symmetric superspace.

4.14.6 The Exceptional RSSS $\text{D}(2, 1; \alpha)/\text{SO}(2) \times \text{SOSp}(2|2)$

We also give one example of a family of Riemannian symmetric superspaces not occurring in the Zirnbauer list. The corresponding infinitesimal objects are taken from the tables of Serganova.

The exceptional Lie superalgebra $\mathfrak{g} = \mathfrak{d}(2, 1; \alpha)$, where $\alpha \in \mathbf{R} \setminus \{0, 1\}$ is defined as follows: The even and odd part of \mathfrak{g} are

$$\mathfrak{g}_0 = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2),$$

$$\mathfrak{g}_1 = \mathbf{R}^2 \otimes \mathbf{R}^2 \otimes \mathbf{R}^2,$$

with the \mathfrak{g}_0 -module structure given by the threefold tensor product of the standard representation of $\mathfrak{sl}(2)$ on \mathbf{R}^2 . The dependence on the parameter α is hidden in the remaining part of the Lie bracket, $\mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_0$, see [Sche 1979], Example 5 of Chapter 1, §1: Let $\psi : \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathbf{R}$ be the non-degenerate skew-symmetric bilinear form given by $\psi(e_1, e_2) = 1$, where $\{e_1, e_2\}$ is the standard basis of \mathbf{R}^2 . Let $P : \mathbf{R}^2 \times \mathbf{R}^2 \rightarrow \mathfrak{sl}(2)$ be the $\mathfrak{sl}(2)$ -invariant bilinear mapping given by

$$P(u, v)w = \psi(v, w)u - \psi(w, u)v$$

for $u, v, w \in \mathbf{R}^2$. Then we define

$$\begin{aligned} [u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3] = & (\sigma_1 \psi(u_2, v_2) \psi(u_3, v_3) P(u_1, v_1), \\ & \sigma_2 \psi(u_1, v_1) \psi(u_3, v_3) P(u_2, v_2), \\ & \sigma_3 \psi(u_1, v_1) \psi(u_2, v_2) P(u_3, v_3)), \end{aligned}$$

where the σ_i are some real numbers not equal to zero depending on α and satisfying $\sigma_1 + \sigma_2 + \sigma_3 = 0$.

We define an involutive automorphism of \mathfrak{g} by

$$\sigma = (\tau \oplus \tau \oplus \text{id}_{\mathfrak{sl}(2)}) \oplus (J_1 \otimes J_1 \otimes \text{id}_{\mathbf{R}^2}),$$

where $\tau : \mathfrak{sl}(2) \rightarrow \mathfrak{sl}(2)$ is defined by $\tau(A) = -A^t$ and $J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The \mathfrak{k} -part of the corresponding decomposition $\mathfrak{d}(2, 1; \alpha) = \mathfrak{k} \oplus \mathfrak{p}$ is $\mathfrak{so}(2) \oplus \mathfrak{osp}(2|2)$.

Let $\text{D}(2, 1; \alpha)$ be the Lie supergroup given by the Harish-Chandra pair

$$(\text{SL}(2) \times \text{SL}(2) \times \text{SL}(2), \mathfrak{d}(2, 1; \alpha)),$$

where the adjoint representation is the standard one. The automorphism σ clearly is induced by an automorphism of $\text{D}(2, 1; \alpha)$ turning

$$(\text{D}(2, 1; \alpha), \text{SO}(2) \times \text{SOSp}(2|2))$$

into a symmetric pair.

The invariant metric is constructed as follows: Let $\langle \cdot, \cdot \rangle_1 : \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathbf{R}$ be the non-degenerate $\text{ad}_{\mathfrak{g}_0}$ -invariant skew-symmetric bilinear form given by

$$\langle u_1 \otimes u_2 \otimes u_3, v_1 \otimes v_2 \otimes v_3 \rangle_1 := \psi(u_1, v_1) \psi(u_2, v_2) \psi(u_3, v_3);$$

by Proposition 4.21, it extends to an $\text{ad}_{\mathfrak{g}}$ -invariant scalar superproduct $\langle \cdot, \cdot \rangle$ on \mathfrak{g} . Since $\langle \sigma(u), \sigma(v) \rangle_1 = \langle u, v \rangle_1$ for all $u, v \in \mathfrak{g}_1$ and since σ is an automorphism, it follows that σ is orthogonal with respect to $\langle \cdot, \cdot \rangle$ on the whole of \mathfrak{g} . Consequently, \mathfrak{k} is orthogonal to \mathfrak{p} as these spaces are eigenspaces of σ . Restricting σ to \mathfrak{p} , we get an $\text{Ad}_{\text{SO}(2) \times \text{SOSp}(2|2)}$ -invariant scalar superproduct which induces an invariant metric on $\text{D}(2, 1; \alpha)/\text{SO}(2) \times \text{SOSp}(2|2)$ turning it into a Riemannian symmetric superspace with underlying manifold $\text{SL}(2)/\text{SO}(2) \times \text{SL}(2)/\text{SO}(2)$.

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